이학박사 학위논문

Modeling of Commodity Index and Relative Derivatives

(원자재 지수와 관련 파생상품의 모델링)

2012년 8월

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이 논문을 이학박사 학위논문으로 제출함

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Modeling of Commodity Index and Relative Derivatives

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Abstract

This thesis suggests two kinds of modeling about commodity indices.

In the first model, we assume that the contract weights of the underlying futures of a commodity index is rebalanced as soon as the futures prices change. This model shows that the value process of commodity index is represented as a martingale and lognormal process under risk neutral measure. In this model, the monetary proportion of each underlying commodity futures for commodity index value satisfies each own target weights(or the index weight) at all times. From this modeling we can take the PDEs and the pricing formulas for the value of European options on commodity indices.

In the second model, we assume that the contract weights of the underlying futures of commodity index is rebalanced periodically. This model is represented as a typical 'stochastic volatility model'. From the second modeling, we can get the PDEs for the value of European options on commodity indices over piecewise time interval i.e. non roll periods and roll periods.

Key words: Commodity Index, Commodity Futures, Commodity Index Modeling, Commodity Index Options, Index Option Price, Hedging

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Chapter 1

Introduction

A Commodity index is similar to portfolio consisting of various commodity futures belong to the segments of agriculture, energy and metal. A Commodity index is well known because of its advantages that the index has hedging effect for inflation and that investing in a commodity index is not different from diversified investment for various commodities.

Ankrim and Hensel(1993)[5] analyzed and compared returns of commodity indices with returns of stock indices and bonds. They said that commodity indices inherit the benefit of real commodity assets and hedge inflation, therefore we expect diversified investment effect from the portfolio consisting of commodity indices and traditional assets. Greer(2000)[15] research the characteristics of commodity index return and announced that there is a negative correlation between the commodity indices and traditional assets during unexpected inflation. And Erb and Harvey(2005)[4] told that the returns of commodity portfolio are made up of cash returns, excess returns and diversification returns. Besides various papers research the characteristics and benefits of commodity indices as an investment vehicle.

Actually there has been increasing interest in the market related to the commodity indices. Even now the commodity indices and relative derivatives such as S&P GSCI, DJ-UBS, RICI and so on are actively traded in global financial market. Nevertheless there is not much the study trying to model the structure of commodity index. This Thesis will attempt modeling of commodity index subject to RICI.

RICI is the commodity index according to the *index weights* designated by the RICI committee. Index weight is the monetary proportion that specific commodity futures occupies within the value of commodity index. This value is assigned to each commodity futures composing RICI. Then we have to calculate the respective shares of commodity futures contracts satisfying the index weights at some time. From these calculation, we can get the proportion of shares of respective commodity futures contracts for the shares of whole commodity futures contracts. these proportions are named *contracts weights*. In fact, if one hold these contracts weights unvaryingly, the index weights are not maintained.

There is an important characteristic of commodity indices. That is the futures prices as the underlying of a commodity index is connected with the specific delivery dates. So, underlying futures have to be rolled to the futures with farther delivery date to define the value of commodity index persistingly. Therefore most commodity indices have rollover periods in which the first nearby futures are rolled to the second nearby futures, roughly speaking. Thus the objects that a commodity index tracks are changed monthly.

However because the futures prices of first and second nearby contracts respectively fluctuate, the respective contracts weights of first and second futures contracts to maintain the index weight are different. So, in the case of RICI, the contracts weights of the second nearby contracts are calculated at the initial time of a rollover period. This process is called *rebalancing*.

These characters are the factors that make mathematical approach for commodity indices hard. Actually the most research about commodity indices are statistical data analysis or empirical studies, but the approaches by mathematical modeling for commodity indices seem to be scarce. This thesis makes attempts to describe the stochastic dynamics of commodity indices in two ways and to derive the equations and the formulas for the prices of derivatives based on the commodity index.

In Chapter 2, first we will suggest the theoretical model about commodity indices. In this model, the portfolio of the futures consisting the commodity index is rebalanced continuously to adjust the given the index weight of each components of the index. Then the monetary proportion of each underlying commodity futures for commodity index value satisfies each own the index weight at all times. Furthermore, we will advert the correlation issue between the futures prices over our first commodity index model, and consider the relation between the original commodity index and a sub-index.

In Chapter 3, we will derive the partial differential equations for the prices of the options on the commodity indices. From these equations we will calculate the price formulas of the various options. Also, we will deal with the sensitivity of the option prices using the formulas of European option prices.

In Chapter 4, we will suggest the second modeling about commodity indices. Because this model is based the structure of RICI, the modeling is more practical than the first modeling in Chapter 2. In the second model, the contracts weights of the components futures of the commodity index is rebalanced periodically to adjust the given target weight. Here, we will treat the second commodity index model as a stochastic volatility model.

Chapter 2

The Model for Commodity Index

2.1 The Model for Commodity Index with Continuously Rebalanced Contract Weights

We will make an attempt to model commodity index in this section. The approach introduced in this section is somewhat theoretical. First of all, let us see the idea of this construction through an example.

First, we will consider the portfolio which is composed of the futures for two distinct commodities, X and Y. Let us assume that Their monetary proportions are 1:1 within the value of portfolio. The value process of this portfolio satisfies the self-financing condition. And assume that if the futures prices change, this portfolio is adjusted to preserve the monetary proportions of X and Y within the value of the portfolio.

Now, let us see the example.

	1st day	2nd day	3rd day
X	1	3	2
Y	2	4	3
Portfolio Value	100	250	$250\times\frac{17}{24}$
Shares of X	50	$\frac{125}{3}$	$125\times\frac{17}{48}$
Shares of Y	25	$\frac{125}{4}$	$125\times\frac{17}{72}$

Table 2.1: The portfolio strategy according to the changes of the futures prices

In the first day, the futures prices of X and Y are 1 and 2 respectively. The initial capital for investing to the portfolio is given as 100. Because

$$100 = 50 + 50 = \frac{50}{1} \times 1 + \frac{50}{2} \times 2$$

Namely, the portfolio is composed of 50-shares futures X and 25-shares futures Y.

Next day, the futures prices X and Y change to 3 and 4. Then the value of the portfolio changes to

$$50 \times 3 + 25 \times 4 = 150 + 100 = 250.$$

Here the ratio of X and Y within 250 is $150 : 100 = 3 : 2$, that is, the ratio 1:1 of X and Y breaks. Because

$$250 = 125 + 125 = \frac{125}{3} \times 3 + \frac{125}{4} \times 4,$$

Adjusting the shares of X and Y to 125/3 and 125/4 respectively, the ratio of X and Y becomes 1:1 again.

In the 3rd day, the futures prices X and Y change to 2 and 3 respectively. Then the value of the portfolio become

$$\frac{125}{3} \times 2 + \frac{125}{4} \times 3 = 125 \left(\frac{2}{3} + \frac{3}{4} \right) = 250 \times \frac{17}{24} \ .$$

In common with the 2nd day, we have to adjust the portfolio to maintain the ratio of X and Y as 1:1. Because

$$250 \times \frac{17}{24} = 125 \times \frac{17}{24} + 125 \times \frac{17}{24} = 125 \times \frac{17}{48} \times 2 + 125 \times \frac{17}{72} \times 3,$$

the portfolio is changed to $\left(125 \times \frac{17}{48},\ 125 \times \frac{17}{72}\right)$.

Next, let us consider the commodity index consisting of the futures prices X and Y which tracks the above portfolio strategy.

Assumption 1.

- Each proportion of X and Y within the value of index is maintained as $\frac{1}{2}$ always. These rates are called the index weights.

- m_X and m_Y are the respective proportion of shares of X and Y. For example, if the shares of X and Y are 2 and 3, then $m_X = \frac{2}{5}$ and $m_Y = \frac{3}{5}$. (Here, 2 and 3 is not important.) These values are adjusted daily after the futures prices change. These weights are called the contract weights. These are defined as follows.

$$m_X = \frac{\frac{1}{2} \times \frac{1}{X}}{\frac{1}{2} \times \frac{1}{X} + \frac{1}{2} \times \frac{1}{Y}} \tag{2.1.1}$$

$$m_Y = \frac{\frac{1}{2} \times \frac{1}{Y}}{\frac{1}{2} \times \frac{1}{X} + \frac{1}{2} \times \frac{1}{Y}} \tag{2.1.2}$$

  So, $0 < m_X < 1,\ 0 < m_Y < 1$ and $m_X + m_Y = 1$ if $X > 0$ and $Y > 0$.

- The value of commodity index is determined as follow at $(t+1)$-th day.

$$I(t+1) = I(t)\frac{m_X X(t+1) + m_Y Y(t+1)}{m_X X(t) + m_Y Y(t)}$$

	1st day	2nd day	3rd day
X	1	3	2
Y	2	4	3
Index Value	100	250	$250\times\frac{17}{24}$
m_X	$\frac{2}{3}$	$\frac{4}{7}$	$\frac{3}{5}$
m_Y	$\frac{1}{3}$	$\frac{3}{7}$	$\frac{2}{5}$

Table 2.2: The index value process according to the changes of the futures prices

In the first day, the initial value of the commodity index is given as 100. By the definition of the contract weights,

$$m_X = \frac{2}{3},\ m_Y = \frac{1}{3}.$$

In the 2nd day, the futures prices X and Y are changed to 3 and 4 respectively. So by the Assumption 1, the index value is

$$100 \cdot \frac{\frac{2}{3}\times 3 + \frac{1}{3}\times 4}{\frac{2}{3}\times 1 + \frac{1}{3}\times 2} = 250.$$

And we have to change the contract weights to maintain the index weights. In other words

$$m_X = \frac{\frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{4}} = \frac{4}{7},$$
$$m_Y = \frac{\frac{1}{2}\times\frac{1}{4}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{4}} = \frac{3}{7}.$$

And in the 3re day, the futures prices X and Y are changed to 2 and 3 respectively. Thus the index value become

$$250 \times \frac{\frac{4}{7}\times 2 + \frac{3}{7}\times 3}{\frac{4}{7}\times 3 + \frac{3}{7}\times 4} = 250 \times \frac{17}{24}$$

and the contract weights are changed to $\dfrac{3}{5}$ and $\dfrac{2}{5}$ for X and Y respectively.

Therefore the commodity index defined like this process tracks the portfolio strategy described in Table 2.1. In other words, we can say the portfolio strategy in Table 2.1 replicates the commodity index in Table 2.1.

Now in the next subsections, we will describe stochastic model for commodity index with rebalanced continuously contract weights.

2.1.1 Non Roll Periods

Glossary

In this subsection we will describe the stochastic model for commodity index during a non roll period.

- $X(t,T),\ Y(t,T)$: the futures prices of first nearby contracts
- $w_x,\ w_y$: index weights satisfying $0 < w_x < 1,\ 0 < w_y < 1$ and $w_x + w_y = 1$.
- $m_x(t,T),\ m_y(t,T)$: the contract weight satisfying the following equations

$$\begin{cases} \frac{m_x(t,T)X(t,T)}{m_x(t,T)X(t,T)+m_y(t,T)Y(t,T)} &= w_x \\ \frac{m_y(t,T)Y(t,T)}{m_x(t,T)X(t,T)+m_y(t,T)Y(t,T)} &= w_y \\ m_x(t,T)+m_y(t,T) &= 1 \end{cases} . \tag{2.1.3}$$

Then the values of $m_x(t,T)$ and $m_y(t,T)$ are

$$\begin{aligned} m_x(t,T) &= \frac{\frac{w_x}{X(t,T)}}{\frac{w_x}{X(t,T)}+\frac{w_y}{Y(t,T)}} \\ m_y(t,T) &= \frac{\frac{w_y}{Y(t,T)}}{\frac{w_x}{X(t,T)}+\frac{w_y}{Y(t,T)}} \end{aligned}$$

respectively.

Assumption 2. Let us assume that it is defined a d-dimensional Brownian motion $\mathbf{W}^{\mathbb{P}}(t)$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Here $\mathbb{P}$ is a physical measure. And $\{\mathscr{F}_t\}_{0\le t\le\tau}$ is a filtration generated by $\mathbf{W}^{\mathbb{P}}(t)$, moreover $\mathscr{F}_\tau = \mathscr{F}$. Then By Girsanov Theorem there is a d-dimensional Brownian Motion, $\mathbf{W}(t)$ under risk neutral measure $\mathbb{Q}$.

Assume that the dynamics of the futures prices $X(t,T)$, $Y(t,T)$ under risk neutral measure $\mathbb{Q}$ are as follows.

$$\begin{aligned} dX(t,T) &= X(t,T)\sigma_X(t,T)\cdot d\mathbf{W}_t \\ dY(t,T) &= Y(t,T)\sigma_Y(t,T)\cdot d\mathbf{W}_t \end{aligned}$$

where $\sigma_X(t,T), \sigma_Y(t,T)$ are deterministic 2-dimensional vector functions. And we assume that $\mathbf{W}_t$ is a 2- dimensional column vector consisting of two independent Wiener processes.

Our commodity index is based on the values of first nearby commodity futures prices. And the initial price of the commodity index $I(0)$ is given.

Then, the excess return of commodity index is defined as the following formula,

$$\frac{I(t+\Delta t)}{I(t)} = \frac{m_x(t,T)X(t+\Delta t,T) + m_y(t,T)Y(t+\Delta t,T)}{m_x(t,T)X(t,T) + m_y(t,T)Y(t,T)}.$$

Using the equation (2.1.3), $I(t+\Delta t)$ is represented the following equation.

$$I(t+\Delta t) = I(t)\left(\frac{w_x}{X(t,T)}X(t+\Delta t,T) + \frac{w_y}{Y(t,T)}Y(t+\Delta t,T)\right).$$

However

$$
\begin{aligned}
& I(t+\Delta t) - I(t) \\
= \quad & I(t)\left(\frac{w_x}{X(t,T)}X(t+\Delta t,T) + \frac{w_y}{Y(t,T)}Y(t+\Delta t,T)\right) - I(t) \\
= \quad & I(t)\left(\frac{w_x}{X(t,T)}\left(X(t+\Delta t,T) - X(t,T)\right) + \frac{w_y}{Y(t,T)}\left(Y(t+\Delta t,T) - Y(t,T)\right)\right).
\end{aligned}
$$

Letting $\Delta t \to 0$

$$
dI(t) \quad = \quad I(t)\left(\frac{w_x}{X(t,T)}dX(t,T) + \frac{w_y}{Y(t,T)}dY(t,T)\right).
$$

Substituting the SDEs for the futures prices in Assumption 2, the dynamics of our commodity index is as follow.

$$
dI(t) = I(t)\sigma(t,T)\cdot d\mathbf{W}_t
$$

where

$$
\begin{aligned}
\sigma(t,T) \quad &= \quad w_x\sigma_X(t,T) + w_y\sigma_Y(t,T) \\
&= \quad \begin{pmatrix} w_x\|\sigma_X(t,T)\|a + w_y\|\sigma_Y(t,T)\|b \\ w_x\|\sigma_X(t,T)\|\sqrt{1-a^2} + w_y\|\sigma_Y(t,T)\|\sqrt{1-b^2} \end{pmatrix} \qquad (2.1.4)
\end{aligned}
$$

and

$$
ab + \sqrt{1-a^2}\sqrt{1-b^2} = \rho
$$

and ρ is the correlation coefficient of $X(t,T)$ and $Y(t,T)$.

Therefore the dynamic of commodity index $I(t)$ is a martingale and lognormal process during non roll periods under the risk neutral measure.

2.1.2 Roll Periods

In this subsection we will describe the stochastic model for commodity index during a roll period. We will deal with the following two cases in this subsection.

- case 1 : the case that all futures contracts components of the commodity index change the second nearby contracts respectively in a roll period.
- case 2 : the case that some futures contracts components of the commodity index does not change the second nearby contracts in a roll period.

Glossary

- w : the index weight of $X(t,T)$ for an arbitrary T. So the index weight of $Y(t,T)$ is $1-w$.
- $X(t,T), Y(t,T)$: the futures prices of the first nearby contracts
- $X(t,T'), Y(t,T')$: the futures prices of the second nearby contracts
- $I(t)$: the index consisting of the futures prices $X(t,T), Y(t,T)$
- $J(t)$: the index consisting of the futures prices $X'(t,T), Y'(t,T)$
- $I_R(t)$: the value of commodity index during a roll period
- $\alpha(t)$: Suppose that p_0 is the initial time and p_1 is the terminal time of a roll period. Then roll weight $\alpha(t)$ is defined as follow.

$$\alpha(t) = \frac{p_1 - t}{p_1 - p_0} \quad , p_0 \le t \le p_1.$$

In a roll period, the value of commodity index is determined by four futures prices $X(t,T), Y(t,T)$ and $X(t,T'), Y(t,T')$. We conceive the idea that the commodity index $I(t)$ composed with $X(t,T), Y(t,T)$ is rolled to the commodity index $J(t)$ composed with $X(t,T'), Y(t,T')$.

Assumption 3. The stochastic processes of four futures prices $X(t,T)$, $Y(t,T)$, $X(t,T')$ and $Y(t,T')$ are represented as

$$\begin{aligned} dX(t,T) &= X(t,T)\sigma_X(t,T)\cdot d\mathbf{W}_t \\ dY(t,T) &= Y(t,T)\sigma_Y(t,T)\cdot d\mathbf{W}_t \\ dX(t,T') &= X(t,T')\sigma_X(t,T')\cdot d\mathbf{W}_t \\ dY(t,T') &= Y(t,T')\sigma_Y(t,T')\cdot d\mathbf{W}_t \end{aligned}$$

respectively under risk neutral measure. The four sigma volatilities in these equations are deterministic 2-dimensional vector functions. And $\mathbf{W}_t$ is a 2-dimensional vector process.

<u>**Case 1**</u>

Define the excess return of our commodity index during roll periods as follow.

$$\frac{I_R(t+\Delta t)}{I_R(t)} = \frac{\alpha(t)}{I(t)}I(t+\Delta t) + \frac{(1-\alpha(t))}{J(t)}J(t+\Delta t).$$

Then,

$$\begin{aligned} dI_R(t) &= I_R(t)\left(\frac{\alpha(t)}{I(t)}dI_t + \frac{1-\alpha(t)}{J(t)}dJ_t\right) \\ &= I_R(t)\alpha(t)\left(\frac{w}{X(t,T)}dX(t,T) + \frac{1-w}{Y(t,T)}dY(t,T)\right) \\ &\quad + I_R(t)\left(1-\alpha(t)\right)\left(\frac{w}{X(t,T')}dX(t,T') + \frac{1-w}{Y(t,T')}dY(t,T')\right) \end{aligned}$$

Substituting the result of the preceding subsection,

$$\begin{aligned} dI_R(t) &= I_R(t)\alpha(t)\left(w\sigma_X(t,T)\cdot d\mathbf{W}_t + (1-w)\sigma_Y(t,T)\cdot d\mathbf{W}_t\right) \\ &\quad + I_R(t)(1-\alpha(t))\left(w\sigma_X(t,T')\cdot d\mathbf{W}_t + (1-w)\sigma_Y(t,T')\cdot d\mathbf{W}_t\right) \\ &= I_R(t)\sigma_R(t,T,T')\cdot d\mathbf{W}_t \end{aligned}$$

where

$$\sigma_R(t,T,T') = \tag{2.1.5}$$
$$\begin{pmatrix} w\left(\alpha(t)\|\sigma_X(t,T)\|a + (1-\alpha(t))\|\sigma_X(t,T')\|c\right) \\ \quad +(1-w)\left(\alpha(t)\|\sigma_Y(t,T)\|b + (1-\alpha(t))\|\sigma_Y(t,T')\|d\right) \\ w\left(\alpha(t)\|\sigma_X(t,T)\|\sqrt{1-a^2} + (1-\alpha(t))\|\sigma_X(t,T')\|\sqrt{1-c^2}\right) \\ \quad +(1-w)\left(\alpha(t)\|\sigma_Y(t,T)\|\sqrt{1-b^2} + (1-\alpha(t))\|\sigma_Y(t,T')\|\sqrt{1-d^2}\right) \end{pmatrix},$$

$$cd + \sqrt{1-c^2}\sqrt{1-d^2} = \rho'$$

and ρ' is the correlation coefficient of $X(t,T')$ and $Y(t,T')$.

Case 2

Let us now consider the case that some contracts among the futures contracts composing the commodity index do not change to the second nearby contracts during a rollover period. To consider this case in our simple model, we may assume that $I(t)$ is composed by $X(t,T), Y(t,T)$ and $J(t)$ is composed by $X(t,T), Y(t,T')$. Namely, $X(t,T)$ does not change to the second nearby contract in a rollover period. Then the dynamics of $I(t), J(t)$ are

$$\begin{aligned} dI_t &= I(t)\left(\frac{w}{X(t,T)}dX(t,T) + \frac{1-w}{Y(t,T)}dY(t,T)\right), \\ dJ_t &= J(t)\left(\frac{w}{X(t,T)}dX(t,T) + \frac{1-w}{Y(t,T')}dY(t,T')\right) \end{aligned}$$

respectively. Then,

$$\begin{aligned} dI_R(t) &= I_R(t)\left(\frac{\alpha(t)}{I(t)}dI_t + \frac{1-\alpha(t)}{J(t)}dJ_t\right) \\ &= I_R(t)\left(\frac{w}{X(t,T)}dX(t,T) + \frac{\alpha(t)(1-w)}{Y(t,T)}dY(t,T) + \frac{(1-\alpha(t))(1-w)}{Y(t,T')}dY(t,T')\right) \\ &= I_R(t)\sigma_R(t,T,T')\cdot d\mathbf{W}_t \end{aligned}$$

where

$$
\begin{aligned}
& \sigma_R(t,T,T') \\
= \; & w\sigma_X(t,T) + (1-w)\left(\alpha(t)\sigma_Y(t,T) + (1-\alpha(t)\sigma_Y(t,T'))\right) \\
= \; & \begin{pmatrix} w\|\sigma_X(t,T)\|a + (1-w)\left(\alpha(t)\|\sigma_Y(t,T)\|b + (1-\alpha(t))\|\sigma_Y(t,T')\|d\right) \\ w\|\sigma_X(t,T)\|\sqrt{1-a^2} \\ \quad +(1-w)\left(\alpha(t)\|\sigma_Y(t,T)\|\sqrt{1-b^2} + (1-\alpha(t))\|\sigma_Y(t,T')\|\sqrt{1-d^2}\right) \end{pmatrix}.
\end{aligned}
$$

Therefore, in both cases, the commodity index is a martingale and log-normal process under the risk neutral measure during roll periods.

2.2 Correlation Coefficient Issue

Suppose that two kinds of European futures options on $X(t,T)$ and $Y(t,T)$ which are components of the commodity index, and the European commodity index options are tradable. Then we can get calculate the implied volatility of the options and we may take the three absolute values of volatilities for the futures prices $X(t,T)$,$Y(t,T)$ and the commodity index composed of $X(t,T)$ and $Y(t,T)$ [3].

2.2.1 Non Roll periods

As the above result, the volatility of the commodity index is as follows.

$$
\sigma(t,T) = \begin{pmatrix} w_x\|\sigma_X(t,T)\|a + w_y\|\sigma_Y(t,T)\|b \\ w_x\|\sigma_X(t,T)\|\sqrt{1-a^2} + w_y\|\sigma_Y(t,T)\|\sqrt{1-b^2} \end{pmatrix}
$$

Then,

$$
\begin{aligned}
& \|\sigma(t,T)\|^2 \\
= \; & (w_x\|\sigma_X(t,T)\|a + w_y\|\sigma_Y(t,T)\|b)^2 \\
& + \left(w_x\|\sigma_X(t,T)\|\sqrt{1-a^2} + w_y\|\sigma_Y(t,T)\|\sqrt{1-b^2}\right)^2 \\
= \; & (w_x\|\sigma_X(t,T)\|)^2 + (w_y\|\sigma_Y(t,T)\|)^2 + 2w_X w_Y\|\sigma_X(t,T)\|\|\sigma_Y(t,T)\|\rho
\end{aligned}
$$

So, if we can know the absolute value of volatilities i.e. $\|\sigma_X(t,T)\|, \|\sigma_Y(t,T)\|$ and $\|\sigma(t,T)\|$, then we can get the value of the correlation coefficient ρ by the above equation.

2.2.2 Roll Periods

$$
\begin{tikzcd}
X(t,T) \arrow[r, "\lambda_X"] \arrow[d, dash, "\rho_T"'] \arrow[dr, dash, "\eta_{X_T,Y'_T}" near start] & X(t,T') \arrow[d, dash, "\rho_{T'}"] \arrow[dl, dash, "\eta_{Y_T,X'_T}"' near start] \\
Y(t,T) \arrow[r, "\lambda_Y"'] & Y(t,T')
\end{tikzcd}
$$

Figure 2.1: the correlation coefficients for the futures prices during rollover periods

Glossary

- $X(t,T)$ and $Y(t,T)$: the 1st nearby futures prices which are the components of the commodity index.

- $X(t,T)$ and $Y(t,T)$: the 2nd nearby futures prices which will be the components of the commodity index.
- λ_X : the correlation coefficients of $X(t,T)$ and $X(t,T')$.
- λ_Y : the correlation coefficients of $Y(t,T)$ and $Y(t,T')$.
- ρ_T : the correlation coefficients of $X(t,T)$ and $Y(t,T)$.
- $\rho_{T'}$: the correlation coefficients of $X(t,T')$ and $Y(t,T')$.
- η_{X_T,Y'_T} : the correlation coefficients of $X(t,T)$ and $Y(t,T')$.
- η_{Y_T,X'_T} : the correlation coefficients of $X(t,T)$ and $Y(t,T')$.

Suppose that we know the values of the correlation coefficients ρ_T and $\rho_{T'}$. Also existing the futures exchange options on $X(t,T)$ and $X(t,T')$, and getting the prices data of the futures exchange options we may get the values of the correlation coefficients λ_X. So does λ_Y. However we have to determine at least three values ρ_T, λ_X and $\eta_{T,T'}$ among these correlation coefficients to simulate this commodity index. Suppose that the correlation between the two underlying commodity prices, $X(t,T)$ and $Y(t,T)$ is uniform as ρ regardless of T. (Definitely ρ may be a function for the time variable t.) Also assume that we know the ρ and λ_X and that the four futures prices satisfy

$$
\begin{aligned}
dX(t,T) &= X(t,T)\sigma_X(t,T)dW_1 \\
dY(t,T) &= Y(t,T)\sigma_Y(t,T)\left(\rho\, dW_1 + \sqrt{1-\rho^2}\, dW_2\right) \\
dX(t,T') &= X(t,T')\sigma_X(t,T')\left(\lambda_X\, dW_1 + \sqrt{1-\lambda_X^2}\, dW_2\right) \\
dY(t,T') &= Y(t,T')\sigma_Y(t,T')\left(\eta_{T,T'}\, dW_1 + \sqrt{1-\eta_{T,T'}^2}\, dW_2\right)
\end{aligned}
$$

where W_1 and W_2 are the independent Brownian motions. In these equations $\sigma_X(t,T), \sigma_Y(t,T), \sigma_X(t,T'), \sigma_Y(t,T')$ and W_1, W_2 are scalar quantities. Then the following equation holds because the correlation of $X(t,T')$ and $Y(t,T')$ is ρ.

$$\begin{pmatrix} \lambda_X & \sqrt{1-\lambda_X^2} \\ \eta & \sqrt{1-\eta_{T,T'}^2} \end{pmatrix} \begin{pmatrix} \lambda_X & \eta_{T,T'} \\ \sqrt{1-\lambda_X^2} & \sqrt{1-\eta_{T,T'}^2} \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Then

$$\lambda_X \eta_{T,T'} + \sqrt{1-\lambda_X^2}\sqrt{1-\eta_{T,T'}^2} = \rho.$$

And by simple calculations, η satisfies the quadratic equation as follow.

$$\eta_{T,T'}^2 - 2\rho\lambda_X\eta_{T,T'} + \rho^2 + \lambda_X^2 - 1 = 0.$$

Therefore $\eta_{T,T'}$ is taken as follows.

$$\eta_{T,T'} = \rho\lambda_X \pm \sqrt{(1-\rho^2)(1-\lambda_X^2)}.$$

2.3 The Sub-indexes

Before we investigate the relation between the original commodity index and relative sub-index, first we will expand generally the modeling for commodity index.

2.3.1 The Commodity Index Consisting of N Futures Contracts

<u>Glossary</u>

- $X_i(t,T),\ (i = 1, 2, ..., N)$ are the futures prices for N futures contracts consisting the original commodity index.

- $X_i(t,T')$ is the futures price of the second nearby contracts.

- w_i is the index weights of $X_i(t,T)$ for each i which satisfies $0 < w_i < 1$ and $\Sigma_{j=1}^{N}\ w_j = 1$.

- $m_i(t,T)$ is the contracts weight of the i-th futures price for rebalancing. $m_i(t,T)$ satisfies the following equations.

$$\frac{m_i(t,T)X_i(t,T)}{\Sigma_{j=1}^{N}\ m_j(t,T)X_j(t,T)} = w_i \quad \text{for all } i, \tag{2.3.1}$$

$$\Sigma_{j=1}^{N} m_j(t,T) = 1. \tag{2.3.2}$$

  So the values of $m_i(t,T)$ are

$$\frac{w_i/X_i(t,T)}{\Sigma_{j=1}^{N}\ w_j/X_j(t,T)}$$

  for each i.

- $\alpha(t)$: Suppose that p_0 is the initial time and p_1 is the terminal time of a roll period. Then roll weight $\alpha(t)$ is defined as follow.

$$\alpha(t) = \frac{p_1 - t}{p_1 - p_0} \quad (p_0 \le t \le p_1).$$

Assumption 4.

- $X_i(t,T)$ satisfy as following SDE.

$$dX_i = X_i(t,T)\sigma_i(t,T) \cdot d\mathbf{W}_t.$$

for any T. In this equation, $\mathbf{W}_t$ is a d-dimensional stochastic vector process. Its elements are d independent wiener processes. Because the number of the commodities is N, the correlation matrix for the dynamics of the N commodities is $N \times N$ matrix.

- The excess return of commodity index is determined as follow during non roll periods.

$$\frac{I(t+\Delta t)}{I(t)} = \frac{\Sigma_{j=1}^{N}\ m_j(t,T)X_j(t+\Delta t,T)}{\Sigma_{j=1}^{N}\ m_j(t,T)X_j(t,T)}.$$

- The excess return of the commodity index is determined as follow during roll periods.

$$\begin{aligned}\frac{I(t+\Delta t)}{I(t)} &= \alpha(t)\frac{\Sigma_{j=1}^{N}\ m_j(t,T)X_j(t+\Delta t,T)}{\Sigma_{j=1}^{N}\ m_j(t,T)X_j(t,T)} \\ &\quad + (1-\alpha(t))\,\frac{\Sigma_{j=1}^{N}\ m_j(t,T')X_j(t+\Delta t,T')}{\Sigma_{j=1}^{N}\ m_j(t,T')X_j(t,T')}.\end{aligned}$$

Therefore the dynamic of the commodity index $I(t)$ during non roll periods and roll periods are as follows respectively.

$$\begin{aligned}dI(t) &= I(t)\left(\Sigma_{j=1}^{N}\ \frac{w_j}{X_j(t,T)}dX_j\right) \\ &= I(t)\left(\Sigma_{j=1}^{N}\ w_j\sigma_j(t,T)\right)\cdot d\mathbf{W}_t\end{aligned}$$

and

$$\begin{aligned}dI(t) &= I(t)\alpha(t)\left(\Sigma_{j=1}^{N}\ \frac{w_j}{X_j(t,T)}dX_j(t,T)\right) \\ &\quad + I(t)(1-\alpha(t))\left(\Sigma_{j=1}^{N}\ \frac{w_j}{X_j(t,T')}dX_j(t,T')\right) \\ &= I(t)\left(\Sigma_{j=1}^{N} w_j\left(\alpha(t)\sigma_j(t,T)+(1-\alpha(t))\,\sigma_j(t,T')\right)\right)\cdot d\mathbf{W}_t.\end{aligned}$$

Definition 2.3.1. (the commodity index for N futures contracts during non roll periods.) The commodity index $I(t)$ satisfies the following SDE .

$$dI(t) = I(t)\sigma(t,T) \cdot d\mathbf{W}_t. \tag{2.3.3}$$

where

$$\sigma(t,T) = \Sigma_{j=1}^{N} w_j \sigma_j(t,T).$$

Definition 2.3.2. (the commodity index for N futures contracts during roll periods.) The commodity index $I(t)$ satisries the following SDE during roll periods.

$$dI(t) = I(t)\sigma(t,T,T') \cdot d\mathbf{W}.$$

where

$$\sigma(t,T,T') = \Sigma_{j=1}^{N} w_j \left(\alpha(t)\sigma_j(t,T) + (1-\alpha(t))\sigma_j(t,T')\right).$$

2.3.2 The Sub-Index

Assume that the original commodity index has various futures contracts as its component. But we can choose some kinds of those futures contracts to compose a new commodity index. Such commodity index is called the sub-index. Ordinarily the futures contracts of the same kind segment; energy, agriculture or metal, compose the new sub-index.

Glossary

- N: The number of components futures consisting our original commodity index.

- segment X, segment Y and segment Z: The components of the commodity index are classified into these three segments.
- n_X, n_Y and n_Z: the positive integers of futures contracts in respective segments. $n_X + n_Y + n_Z = N$.
- The segment X is composed of the futures prices $X_i(t,T)$ $(i = 1, 2, ..., n_1)$ with the index weights w_i, the segment Y is composed of the futures prices $Y_j(t,T)$ $(j = 1, 2, ..., n_2)$ with the index weights w_j and the segment Z is composed of the futures prices $Z_k(t,T)$ $(k = 1, 2, ..., n_3)$ with the index weights w_k for any T.
- $$T_X = \Sigma_{i=1}^{n_X} w_i, \quad T_Y = \Sigma_{j=1}^{n_Y} w_j, \quad T_Z = \Sigma_{i=1}^{n_Z} w_k.$$

  Namely T_X,T_X and T_X are the total weights of respect segments for the original commodity index. So T_X, T_Y, T_Z are constant and $T_X + T_Y + T_Z = 1$.

Definition 2.3.3. (The sub-index for the segment X) Let us give the futures price $X_i(t,T)$ new sub-index weight $\dfrac{w_i}{T_X}$ for each i.

Then the value process I_X of the sub-index composed by the segment X satisfies the following SDE.

$$dI_X(t) = I_X(t)\sigma_X(t,T) \cdot d\mathbf{W}_t.$$

where

$$\sigma_X(t,T) = \Sigma_{i=1}^{n_X} \frac{w_i}{T_X}\sigma_i(t,T).$$

The other sub-indices are defined like this.

By the Subsection 2.3.1 the dynamic of commodity index is as follow.

$$\begin{aligned}\frac{dI(t)}{I(t)} &= \Sigma_{i=1}^{n_X} \frac{w_i}{X_i(t,T)} dX_i(t,T) \\ &+ \Sigma_{j=1}^{n_Y} \frac{w_j}{Y_j(t,T)} dY_j(t,T) \\ &+ \Sigma_{k=1}^{n_Z} \frac{w_k}{Z_k(t,T)} dZ_k(t,T).\end{aligned}$$

Thus

$$\begin{aligned}\frac{dI(t)}{I(t)} &= T_X \Sigma_{i=1}^{n_X} \frac{w_i}{T_X} \frac{dX_i(t,T)}{X(t,T)} \\ &+ T_Y \Sigma_{j=1}^{n_Y} \frac{w_j}{T_Y} \frac{dY_j(t,T)}{Y(t,T)} \\ &+ T_Z \Sigma_{k=1}^{n_Z} \frac{w_k}{T_Z} \frac{dZ_k(t,T)}{Z(t,T)}\end{aligned}$$

However

$$\begin{aligned}\Sigma_{i=1}^{n_X} \frac{w_i}{T_X} \frac{dX_i(t,T)}{X(t,T)} &= \frac{dI_X}{I_X(t)} \\ \Sigma_{j=1}^{n_Y} \frac{w_j}{T_Y} \frac{dY_j(t,T)}{Y(t,T)} &= \frac{dI_Y}{I_Y(t)} \\ \Sigma_{k=1}^{n_Z} \frac{w_k}{T_Z} \frac{dZ_k(t,T)}{Z(t,T)} &= \frac{dI_Z}{I_Z(t)}.\end{aligned}$$

Therefore

$$\frac{dI(t)}{I(t)} = T_X \frac{dI_X}{I_X(t)} + T_Y \frac{dI_Y}{I_Y(t)} + T_Z \frac{dI_Z}{I_Z(t)} \ .$$

In other words, the return rate of original commodity index is represented as the linear combination of the return rates of all sub-indices which are weighted by total segment weights.

THEOREM 2.3.1. Let I be the value process of the original commodity index and I_X, I_Y and I_Z be the value processes of the sub-indices. Then

$$\frac{dI(t)}{I(t)} = T_X \frac{dI_X}{I_X(t)} + T_Y \frac{dI_Y}{I_Y(t)} + T_Z \frac{dI_Z}{I_Z(t)} \ .$$

And the volatility vector $\sigma(t, T)$ is

$$\sigma(t, T) = T_X \sigma_X(t, T) + T_Y \sigma_Y(t, T) + T_Z \sigma_Z(t, T)$$

where $\sigma_X(t, T)$, $\sigma_Y(t, T)$ and $\sigma_Z(t, T)$ are the volatilities of the respect sub-indices.

Chapter 3

Option Pricing

In this chapter we will derive Black-Scholes PDE for the prices of European options on the commodity index and calculate the prices of various options on the commodity index.

3.1 PDEs for The Commodity Index Option

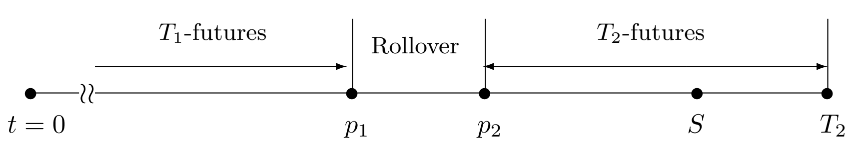

Figure 3.1: underlying futures of the commodity index during the time interval $[0, S]$

Glossary

- S : the maturity of an commodity index option
- T_2 : the delivery date of the underlying futures contracts during the last month which the commodity index option is valid. it is satisfying $T_2 > S$.
- $[p_1, p_2]$: the interval of the last roll period included in the interval $[0, S]$
- $V(t, I)$: the price of the commodity index option

We will induce the partial differential equations of the options on the commodity index in non roll periods and roll periods respectively.

3.1.1 Non Roll Periods

We will consider about the time interval $[p_2, S]$ in this subsection.
π is the value of portfolio consisting of 1-share commodity index option and the futures contracts with futures prices $X(t, T_2), Y(t, T_2)$ which have Δ_X-shares and Δ_Y-shares respectively. So

$$\pi(t) = V(t, I).$$

This is because the value of futures contracts is zero all the time.

Then,

$$
\begin{aligned}
& d\pi(t) \\
= \;& dV + \Delta_X dX(t,T_2) + \Delta_Y dY(t,T_2) \\
= \;& \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial I}dI + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}(dI)^2 + \Delta_X dX(t,T_2) + \Delta_Y dY(t,T_2) \\
= \;& \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial I}\left(\frac{w_X I}{X(t,T_2)}dX + \frac{w_Y I}{Y(t,T_2)}dY\right) \\
& +\frac{1}{2}\frac{\partial^2 V}{\partial I^2}I^2\sigma(t,T_2)\cdot\sigma(t,T_2)dt + \Delta_X dX + \Delta_Y dY.
\end{aligned}
$$

REMARK 3.1.1. (*hedging portfolio during non roll periods*)

$$
\begin{aligned}
\Delta_X &= -\frac{\partial V}{\partial I}\frac{w_X I}{X(t,T_2)}, &\quad (3.1.1)\\
\Delta_Y &= -\frac{\partial V}{\partial I}\frac{w_Y I}{Y(t,T_2)}. &\quad (3.1.2)
\end{aligned}
$$

Then,

$$
d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}I^2\sigma(t,T_2)\cdot\sigma(t,T_2)\right)dt
$$

where $\sigma(t,T_2)$ is defined in the same way as the equation 2.1.4. But because there are no arbitrage opportunities,

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}I^2\sigma(t,T_2)\cdot\sigma(t,T_2) = r\pi = rV
$$

That is

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}I^2\sigma(t,T_2)\cdot\sigma(t,T_2) - rV = 0.
$$

Therefore we have the following lemma by the Feynman-Kač Theorem.

Lemma 3.1.1. Assume that I satisfies the following stochastic differential equation

$$
\begin{cases}
dI(u) &= I(u)\sigma(u;T_2)\cdot d\mathbf{W}_u \\
I(t) &= I
\end{cases},
$$

the process

$$\exp\left(-\int_u^S r(\tau)d\tau\right)\frac{\partial V}{\partial I}(u,I_u)\sigma(u,T_2)$$

is in $\mathscr{L}^2[0,S]$ and V is a solution to the boundary value problem

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\dfrac{\partial^2 V}{\partial I^2}I^2\sigma(t,T_2)\cdot\sigma(t,T_2)-rV &= 0 \\ V(S,I_S) &= \Phi(I_S) \end{cases}$$

where Φ is a S-contingent claim. Then V has the representation

$$V(t,I)=\exp\left(-\int_t^S r(\tau)d\tau\right)E\left[\Phi(I_S)|\mathscr{F}_t\right].$$

where $t\in[p_2,S]$ which is the non roll period ending at time S.

3.1.2 Roll Periods

We will consider about the time interval $p_1\le t<p_2$ in this subsection. Let us define the boundary value in this time interval as follows.

$$\lim_{t\nearrow p_2} V(t,I_R(t))=V(p_2,I_{p_2})$$

And we consider the portfolio consisting of 1-share commodity index option and four kinds of futures contracts with the futures prices $X(t,T_1)$, $Y(t,T_1)$, $X(t,T_2)$ and $X(t,T_2)$ which are Δ_{X_1}, Δ_{Y_1}, Δ_{X_2} and Δ_{Y_2} shares respectively. $X(t,T_1)$, $Y(t,T_1)$ are the futures prices underlying the commodity index just before the time p_1. Let us π be the value of the portfolio, then

$$\pi=V(t,I)$$

Then

$$
\begin{aligned}
& d\pi \\
= \;& dV + \Delta_{X_1} dX(t,T_1) + \Delta_{Y_1} dY(t,T_1) + \Delta_{X_2} dX(t,T_2) + \Delta_{Y_2} dX(t,T_2) \\
= \;& \frac{\partial V}{dt}dt + \frac{\partial V}{\partial I_R} dI_R + \frac{1}{2}\frac{\partial^2 V}{\partial I_R^2}(dI_R)^2 \\
& \quad + \Delta_{X_1} dX(t,T_1) + \Delta_{Y_1} dY(t,T_1) + \Delta_{X_2} dX(t,T_2) + \Delta_{Y_2} dX(t,T_2) \\
= \;& \frac{\partial V}{dt}dt + \frac{1}{2}\frac{\partial^2 V}{\partial I_R^2} I_R^2 \; \sigma_R(t,T_1,T_2)\cdot\sigma_R(t;T_1,T_2)dt \\
& \quad + \frac{\partial V}{\partial I_R}\left(\frac{w\alpha(t)I_R}{X(t,T_1)}dX(t,T_1) + \frac{(1-w)\alpha(t)I_R}{Y(t,T_1)}dY(t,T_1)\right. \\
& \quad + \left.\frac{w(1-\alpha(t))I_R}{X(t,T_2)}dX(t,T_2) + \frac{(1-w)(1-\alpha(t))I_R}{Y(t,T_2)}dY(t,T_2)\right) \\
& \quad + \Delta_{X_1} dX(t,T_1) + \Delta_{Y_1} dY(t,T_1) + \Delta_{X_2} dX(t,T_2) + \Delta_{Y_2} dX(t,T_2).
\end{aligned}
$$

REMARK 3.1.2. (*Hedging Portfolio during roll periods*)

$$\Delta_{X_1} = -\frac{w\alpha(t)I_R}{X(t,T_1)}, \tag{3.1.3}$$

$$\Delta_{Y_1} = -\frac{(1-w)\alpha(t)I_R}{Y(t,T_1)}, \tag{3.1.4}$$

$$\Delta_{X_2} = -\frac{w(1-\alpha(t))I_R}{X(t,T_2)}, \tag{3.1.5}$$

$$\Delta_{Y_2} = -\frac{(1-w)(1-\alpha(t))I_R}{Y(t,T_2)}. \tag{3.1.6}$$

Using this hedging portfolio,

$$d\pi = \left(\frac{\partial V}{dt} + \frac{1}{2}\frac{\partial^2 V}{\partial I_R^2} I_R^2 \; \sigma_R(t;T_1,T_2)\cdot\sigma_R(t,T_1,T_2)\right)dt$$

where $\sigma_R(t,T_1,T_2)$ is defined in the same way as the equation 2.1.5. However, by no arbitrage argument

$$\frac{\partial V}{dt} + \frac{1}{2}\frac{\partial^2 V}{\partial I_R^2} I_R^2 \; \sigma_R(t,T_1,T_2)\cdot\sigma_R(t,T_1,T_2) \quad = \quad rV$$

So, we can get the following conclusion.

Lemma 3.1.2. Assume that I_R satisfies the following stochastic differential equation

$$\begin{cases} dI_R(u) &= I_R(u)\sigma_R(u,T_1,T_2)\cdot d\mathbf{W}_u \\ I_R(t) &= I_R \end{cases},$$

the process

$$\exp\left(-\int_u^{p_2} r(\tau)d\tau\right)\frac{\partial V}{\partial I_R}\left(u,I_R(u)\right)\sigma_R\left(u,T_1,T_2\right)$$

is in $\mathscr{L}^2[0,S]$ and V is a solution to the boundary value problem

$$\begin{cases} \frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial I_R^2}I_R^2\sigma_R(t,T_1,T_2)\cdot\sigma_R(t,T_1,T_2)-rV &= 0 \\ \lim_{t\nearrow p_2}V(t,I_R(t)) &= V(p_2,I_{p_2}) \end{cases}.$$

Then V has the representation

$$V(t,I_R)=\exp\left(-\int_t^{p_2} r(\tau)d\tau\right)E\left[V(p_2,I_{p_2})|\mathscr{F}_{p_2}\right].$$

where $t\in[p_1,p_2)$ which is the last roll period before the maturity of the option.

3.1.3 PDEs for The Commodity Index Option

Definition 3.1.1. The commodity index $\mathbb{I}(t)$ is defined as follows.

$$\mathbb{I}(t)=\begin{cases} I(t) & \text{during non roll periods} \\ I_R(t) & \text{during roll periods} \end{cases}$$

And let us define the volatility of $\mathbb{I}(t)$ as follows.

$$\sigma_{T_t}(t)=\begin{cases} \sigma(t,T_t) & \text{during non roll periods} \\ \sigma(t,T_t,T_t') & \text{during roll periods} \end{cases}$$

where T_t and T_t' are the delivery dates of the first and the second nearby futures contracts at time t respectively.

If the futures prices composing the commodity index is continuous processes and the commodity index is defined like Definition 3.1.1, the index $\mathbb{I}(t)$ is continuous all the time. The volatility for the commodity index is similar.

The underlying futures of commodity index are traded in global exchanges. However the intervals between delivery months of these futures are certain according to specific commodity, and the delivery dates of the underlying futures of commodity index are similar. So T_t and T_t' are satisfy the following relation.

$$T_t' = T_t + \text{constant}.$$

Therefore it is possible the above definition for the volatility of commodity index.

Now, the backward iteration of the previous two steps introduced in this section gives the conclusion as follows.

Theorem 3.1.1. Assume that $\mathbb{I}$ satisfies the following stochastic differential equation

$$\begin{cases} d\mathbb{I}(u) &= \mathbb{I}(u)\sigma_{T_u}(u) \cdot dW_u \\ \mathbb{I}(t) &= \mathbb{I} \end{cases},$$

the process

$$\exp\left(-\int_u^S r(\tau)d\tau\right)\frac{\partial V}{\partial \mathbb{I}}(u, \mathbb{I}_u)\mathbb{I}(u)\sigma_{T_u}(u)$$

is in $\mathbf{L}^2[0, S]$.

Suppose V is a solution to the boundary value problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2} \cdot \dfrac{\partial^2 V}{\partial \mathbb{I}^2}\mathbb{I}^2\sigma_{T_t}(t) \cdot \sigma_{T_t}(t) - rV &= 0 \\ V(t, \mathbb{I}) &= \Phi(\mathbb{I}_S) \end{cases}$$

where Φ is a S-contingent claim.

Then V has the representation

$$V(t,\mathbb{I}) = \exp\left(-\int_t^S r(\tau)d\tau\right) E\left[\Phi(\mathbb{I}_S)|\mathscr{F}_t\right].$$

for all $t \in [0,S]$.

Proof. If $t \in [p_1, p_2)$, then the price of the commodity index call options is

$$V(t,\mathbb{I}) = \exp\left(-\int_t^{p_2} r(\tau)d\tau\right) E\left[V(p_2,\mathbb{I}_{p_2})|\mathscr{F}_{p_2}\right]$$

by the lemma 4.1.2.

$$\begin{aligned} V(t,\mathbb{I}) &= \exp\left(-\int_t^{p_2} r(\tau)d\tau\right) E\left[V(p_2,\mathbb{I}_{p_2})|\mathscr{F}_{p_2}\right] \\ &= \exp\left(-\int_t^{p_2} r(\tau)d\tau\right) E\left[\exp\left(-\int_{p_2}^{S} r(\tau)d\tau\right) E\left[\Phi(\mathbb{I}_S)|\mathscr{F}_t\right]|\mathscr{F}_{p_2}\right] \end{aligned}$$

But assuming the interest rate $r(t)$ is a deterministic function for all t, then

$$\begin{aligned} V(t,\mathbb{I}) &= \exp\left(-\int_t^{S} r(\tau)d\tau\right) E\left[E\left[\Phi(\mathbb{I}_S)|\mathscr{F}_t\right]|\mathscr{F}_{p_2}\right] \\ &= \exp\left(-\int_t^{S} r(\tau)d\tau\right) E\left[\Phi(\mathbb{I}_S)|\mathscr{F}_t\right] \end{aligned}$$

The last equality is because of so called the iterated condition of conditional expectation. And iterating such backward calculation, the result of this theorem holds. □

3.2 Pricing Formula of European Options on The Commodity Index

3.2.1 European Call Options

Assumption 5. From now on, we will use the following notations and assumptions to the end of this chapter.

- The notation $\mathbb{I}(t)$ used in the previous section change to the notation $I(t)$.
- $\sigma(t) := \|\sigma_{T_t}(t)\|$.
- Wiener process W_t in the stochastic differential equation of $I(t)$ is 1-dimension in the Sections 3.2, 3.3, 3.4. Thus, the dynamic of the commodity index $I(t)$ is represented as follows.

$$dI(t) = I(t)\sigma(t)dW_t$$

  where $\sigma(t)$ is deterministic.
- $D(t_1, t_2) = \exp\left(-\int_{t_1}^{t_2} r(\tau)d\tau\right)$ where $r(\tau)$ is a risk free short rate. Namely, $D(t_1, t_2)$ is a discount factor.

By Assumption 5, the value of the commodity index is represented as follows.

$$I(t) = I(0)\exp\left(\int_0^t \sigma(u)dW_u - \frac{1}{2}\int_0^t \sigma^2(u)du\right)$$

Definition 3.2.1. Assume that $\sigma(u)$ is deterministic. Let us define the stochastic process X_T as follow.

$$X_T = \int_t^T \sigma(u)dW_u.$$

It is well-known that X_T is a normal distribution with mean zero and variance $\int_t^T \sigma(u)^2 du$.

LEMMA 3.2.1. $0 \leq t < S$ and $I(t) = I$. Then

$$K \leq I_S$$

if and only if

$$\log\frac{K}{I} + \frac{1}{2}\int_t^S \sigma(t)^2 du \leq X_S.$$

Proof.

$$
\begin{aligned}
& K \leq I_S \\
\Leftrightarrow \quad & K \leq I \exp\left(\int_t^S \sigma(u) dW_u - \frac{1}{2}\int_t^S \sigma^2(u) du\right) \\
\Leftrightarrow \quad & \frac{K}{I} \leq \exp\left(\int_t^S \sigma(u) dW_u - \frac{1}{2}\int_t^S \sigma^2(u) du\right) \\
\Leftrightarrow \quad & \log\frac{K}{I} + \frac{1}{2}\int_t^S \sigma^2(u) du \leq \int_t^S \sigma(u) dW_u = X_S.
\end{aligned}
$$

$\square$

Now, let us calculate the pricing formula of the European call option.

THEOREM 3.2.1 (the pricing formula of European call options on the commodity index). Suppose that the commodity index I satisfies the following stochastic differential equation,

$$dI(u) = I(u)\sigma(u) dW_t$$

where $\sigma(u)$ is a deterministic function. And there is an European call option on the commodity index with the strike price K and the maturity S. Then, the pricing formula of this option is as follow.

$$C(t, I) \quad = \quad D(t, S)\left[IN(d_1) - KN(d_2)\right]$$

where

$$d_1 = \frac{\log\frac{I}{K} + \frac{1}{2}\int_t^S \sigma(u)^2 du}{\sqrt{\int_t^S \sigma(u)^2 du}}$$

and

$$d_2 = d_1 - \sqrt{\int_t^S \sigma(u)^2 du}.$$

REMARK 3.2.1. Considering the pricing formula of futures call options, its form is similar with the pricing formula of the commodity index call options. In other words, the futures value is replaced by the commodity index value in the pricing formula of futures call options.

Proof. First of all, let us define as follows.

$$a = \log \frac{K}{I} + \frac{1}{2}\int_t^S \sigma^2(u)du,$$

$$\sigma_{X_S}^2 = \int_t^S \sigma(u)^2 du.$$

By the theorem 3.1.1,

$$\begin{aligned} C(t,I) &= D(t,S)E\left[(I_S - K)^+ | \mathscr{F}_t\right] \\ &= D(t,S)E\left[\left(Ie^{X_S - \frac{1}{2}\sigma_{X_S}^2} - K\right)^+ | \mathscr{F}_t\right]. \end{aligned}$$

By the lemma 3.2.1,

$$\begin{aligned} C(t,I) &= D(t,S)\int_a^\infty \left(I\exp\left(x - \frac{1}{2}\sigma_{X_S}^2\right) - K\right)\frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma_{X_S}^2}\right)dx \\ &= D(t,S)\int_a^\infty I\exp\left(x - \frac{1}{2}\sigma_{X_S}^2\right)\frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma_{X_S}^2}\right)dx \\ &\quad -D(t,S)K\int_a^\infty \frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma_{X_S}^2}\right)dx \end{aligned}$$

In righthand side of this equation, let us calculate the two integration respectively.

(i)

$$
\begin{aligned}
& \int_a^\infty I \exp\left(x - \frac{1}{2}\sigma_{X_S}^2\right) \frac{1}{\sigma_{X_S}\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma_{X_S}^2}\right) dx \\
= & \frac{I}{\sigma_{X_S}\sqrt{2\pi}} \int_a^\infty \exp\left(-\frac{x^2 - 2\sigma_{X_S}^2 x + \sigma_{X_S}^4}{2\sigma_{X_S}^2}\right) dx \\
= & \frac{I}{\sigma_{X_S}\sqrt{2\pi}} \int_a^\infty \exp\left(-\frac{(x - \sigma_{X_S}^2)^2}{2\sigma_{X_S}^2}\right) dx
\end{aligned}
$$

Using the integration by substitution,

$$
\begin{aligned}
= & \frac{I}{\sqrt{2\pi}} \int_{(a-\sigma_{X_S}^2)/\sigma_{X_S}}^\infty \exp\left(-\frac{y^2}{2}\right) dy \\
= & I \text{ Prob.} \left(Y \geq \frac{a - \sigma_{X_S}^2}{\sigma_{X_S}}\right).
\end{aligned}
$$

Since Y is a standard normal distribution

$$
\begin{aligned}
= & IN\left(\frac{-a + \sigma_{X_S}^2}{\sigma_{X_S}}\right) \\
= & IN\left(\frac{\log \frac{I}{K} + \frac{1}{2}\int_t^S \sigma(u)^2 du}{\sqrt{\int_t^S \sigma(u)^2 du}}\right) \\
= & N(d_1)
\end{aligned}
$$

where

$$
d_1 = \frac{\log \frac{I}{K} + \frac{1}{2}\int_t^S \sigma(u)^2 du}{\sqrt{\int_t^S \sigma(u)^2 du}}
$$

and N is the cumulative distribution function of standard normal distribution.

(ii)

$$\begin{aligned}
& K \int_a^\infty \frac{1}{\sigma_{X_S}\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma_{X_S}^2}\right) dx \\
= \; & K \int_{\frac{a}{\sigma_{X_S}}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\
= \; & K \text{ Prob.} \left(Y \geq \frac{a}{\sigma_{X_S}}\right) \\
= \; & KN\left(-\frac{a}{\sigma_{X_S}}\right) \\
= \; & KN(d_2)
\end{aligned}$$

where

$$d_2 = \frac{\log \frac{I}{K} - \frac{1}{2}\int_t^S \sigma(u)^2 du}{\sqrt{\int_t^S \sigma(u)^2 du}}$$

Therefore, the conclusion of this theorem is derived. □

3.2.2 European Put Options

PROPOSITION 3.2.1. Suppose that an European call option and an European put option on the commodity index I have same maturity S and same strike price K. Let $C(S, I_S)$ and $P(S, I_S)$ be the contingent claims of the call and put options respectively. Then

$$C(S, I_S) - P(S, I_S) = I_S - K.$$

By Proposition 3.2.1, the contingent claim of the put option is

$$P(S, I_S) = C(S, I_S) - I_S + K.$$

Let us consider European call option on the commodity index with the strike price 0 and the maturity S. Then the contingent claim of the option is I_S if the values of the commodity index is positive all the time. Thus by Theorem 3.1.1 the price of the option at time t is

$$D(t,S)E\left[I_S|\mathscr{F}_t\right].$$

So we can get the following Lemma.

THEOREM 3.2.2 (the pricing formula of European put options on the commodity index). Suppose there is an European put option on the commodity index with the maturity S and the strike price K. Then at time t, the price of the put option is

$$P(t,I) = D(t,S)\left(KN(-d_2) - IN(-d_1)\right)$$

where

$$d_1 = \frac{\log\frac{I}{K} + \frac{1}{2}\int_t^S \sigma(u)^2 du}{\sqrt{\int_t^S \sigma(u)^2 du}}$$

and

$$d_2 = d_1 - \sqrt{\int_t^S \sigma(u)^2 du}.$$

Proof. By Theorem 3.1.1

$$\begin{aligned}
P(t,I) &= D(t,S)E\left[P(S,I_S)|\mathscr{F}_t\right] \\
&= D(t,S)E\left[C(S,I_S) - I_S + K|\mathscr{F}_t\right] \\
&= D(t,S)\left[E\left[C(S,I_S)|\mathscr{F}_t\right] - E[I_S|\mathscr{F}_t] + K\right] \\
&= D(t,S)\left[IN(d_1) - KN(d_2) - I + K\right] \\
&= D(t,S)\left[KN(-d_2) - IN(-d_1)\right].
\end{aligned}$$

□

From the proof of Theorem 3.2.2 we can take the following corollary.

COROLLARY 3.2.1 (Put-Call Parity). Assume that there are European call and put options on the commodity index with same maturity and same strike price K. Let $C(t, I)$ and $P(t, I)$the prices of the call and put options respectively. Then

$$C(t, I) - P(t, I) = D(t, S)(I - K).$$

3.3 Pricing Formula of The Digital Options on The Commodity Index

3.3.1 Digital Put Options

Definition 3.3.1. The S-contingent claim of digital put options on the commodity index is defined as follow.

$$\Phi(I_S) = \begin{cases} 0 & , I_S \geq K \\ 1 & , I_S < K \end{cases}$$

Let $P_d(t, I)$ be the value of digital put options.

THEOREM 3.3.1. Assume the commodity index has the following dynamic.

$$dI(u) = I(u)\sigma(u)dW_u.$$

where $\sigma(u)$ is a deterministic function.

Then

$$P_d(t, I) = D(t, S)N(-d_2)$$

where

$$d_2 = \frac{\log \frac{I}{K} - \frac{1}{2}\int_t^S \sigma(u)^2 du}{\sqrt{\int_t^S \sigma(u)^2 du}}.$$

Proof. Let us use the following notation again.

$$a = \log \frac{K}{I} + \frac{1}{2} \int_t^S \sigma^2(u) du.$$

By the theorem 3.1.1.

$$\begin{aligned} P_d(t, I) &= D(t, S) E \left[1_{\{I_S < K\}} \right] \\ &= D(t, S) \text{ Prob.} \left(I_S < K \right). \end{aligned}$$

By the lemma 3.2.1.

$$\begin{aligned} &= D(t, S) \text{ Prob.}(X_S < a) \\ &= D(t, S) N(-d_2) \end{aligned}$$

because

$$\text{Var}(X_S) = \int_t^S \sigma(u)^2 du.$$

□

3.3.2 Digital Call Options

Definition 3.3.2. The S-contingent claim of digital call options on the commodity index is defined as follow.

$$\Phi(I_S) = \begin{cases} 1 & , I_S \geq K \\ 0 & , I_S < K \end{cases}$$

Let $C_d(t, I(t))$ be the value of digital call options.

THEOREM 3.3.2. Assume the commodity index has the following dynamic.

$$dI(u) = I(u) \sigma(u) dW_u.$$

where $\sigma(u)$ is a deterministic function.

Then

$$C_d(t,I) = D(t,S)N(d_2)$$

where

$$d_2 = \frac{\log\frac{I}{K} - \frac{1}{2}\int_t^S \sigma(u)^2 du}{\sqrt{\int_t^S \sigma(u)^2 du}}.$$

Proof. By the theorem 3.1.1.

$$\begin{aligned}
C_d(t,I) &= D(t,S)E[1_{\{I_S \geq K\}}] \\
&= D(t,S) \text{ Prob.}(I_S \geq K) \\
&= D(t,S) \text{ Prob.}(X_S \geq a) \\
&= D(t,S) \text{ Prob.}\left(\frac{X_S}{\sigma_{X_S}} \geq \frac{a}{\sigma_{X_S}}\right) \\
&= D(t,S) \text{ Prob.}\left(Y_S \geq \frac{a}{\sigma_{X_S}}\right) \\
&= D(t,S)N\left(-\frac{a}{\sigma_{X_S}}\right) \\
&= D(t,S)N(d_2)
\end{aligned}$$

where

$$\sigma_{X_S}^2 = \int_t^S \sigma(u)^2 du,$$

and Y is the standard normal distribution. □

REMARK 3.3.1.

$$C_d(t,I) + P_d(t,I) = D(t,S)$$

In other word, the value of portfolio consisting of digital call option and put option which are same strike price and same maturity is equal the value of risk free zero coupon bond.

3.4 Pricing Formula of Barrier Option Price

In this section, we will calculate the price of Barrier options on our commodity index.

Definition 3.4.1.

$$Z_t = \int_0^t \sigma(u)dW_u$$

$$X_t = \int_0^t \sigma(u)dW_u - \frac{1}{2}\int_0^t \sigma^2(u)du$$

Then $I(t) = \exp(X_t)$ holds By the Definition 4.4.1. Be careful that the definition of stochastic process X_t is distinct from the meaning in the Sections 3.2, 3.3.

Definition 3.4.2. For a stochastic process X,

$$M_t^X = \max{}_{0\leq s\leq t}\, X_t$$

The outline for the pricing of barrier options is as follow.

(i) Induce the joint probability measure of (Z, M^Z) from the reflection principle and derive the joint density function of (Z, M^Z).

(ii) Using the Girsanov Theorem, calculate the joint probability measure and the joint density function of (X, M^X).

(iii) calculate the price of barrier options.

LEMMA 3.4.1. Suppose that

$$dZ_u = \sigma(u)dW_u,$$

$$M_t^Z = \max{}_{0\leq s\leq t}\, Z_t.$$

Then for arbitrary $a > 0,\ h > 0$,

$$\begin{aligned} P(M_t^Z \geq a) &= 2P(Z_t \geq a) \\ P(M_t^Z \geq a, Z_t \geq a+h) &= P(M_t^Z \geq a, Z_t \leq a-h) \end{aligned}$$

Proof. Let $\tau = \inf\{t > 0; Z_t = a\}$ and

$$\tilde{Z}_t = \begin{cases} Z_t & , t \leq \tau \\ 2a - Z_t & , t \geq \tau \end{cases}.$$

Then Z_t and $\tilde{Z}_t$ have the same distributions.

But $\{M_t^Z \geq a\} = \{Z_t \geq a\} \sqcup \{Z_t < a,\ M_t^Z \geq a\}$.

And since $\{Z_t < a,\ M_t^Z \geq a\} = \{\tilde{Z}_t > a\}$, therefore

$$\begin{aligned} P(M_t^Z \geq a) &= P(Z_t \geq a) + P(\tilde{Z}_t > a) \\ &= P(Z_t \geq a) + P(Z_t > a) \\ &= 2P(Z_t \geq a). \end{aligned}$$

Furthermore,

$$\begin{aligned} P(M_t^Z \geq a, Z_t \geq a+h) &= P(M_t^Z \geq a, \tilde{Z}_t \geq a+h) \\ &= P(M_t^Z \geq a, 2a - Z_t \geq a+h) \\ &= 2P(M_t^Z \geq a, Z_t \leq a-h). \end{aligned}$$

□

LEMMA 3.4.2. Suppose that

$$dZ_u = \sigma(u) dW_u,$$

$$M_t^Z = \max{}_{0 \leq s \leq t} Z_t.$$

Then the joint density function of (Z_t, M_t^Z) on $\{(a,b) : a \leq b,\ b \geq 0\}$ is

$$f_{Z_t, M_t^Z}(a,b) = \sqrt{\frac{2}{\pi}} \cdot \frac{2b-a}{\sigma_{Z_t}^3} \exp\left(-\frac{(2b-a)^2}{2\sigma_{Z_t}^2}\right)$$

where

$$\begin{aligned} \sigma_{Z_t}^2 &= \mathrm{Var}(Z_t) \\ &= \int_0^t \sigma(t)^2 du. \end{aligned}$$

Proof.

$$\begin{aligned} P(Z_t < a,\ M_t^Z \geq b) &= P(Z_t < b-(b-a),\ M_t^Z \geq b) \\ &= P(Z_t > b+(b-a),\ M_t^Z \geq b) \\ &= P(Z_t > 2b-a) \\ &= \int_{2b-a}^{\infty} \frac{1}{\sigma_{Z_t}\sqrt{2\pi}} \exp\left(-\frac{z^2}{2\sigma_{Z_t}}\right) dz \end{aligned}$$

But since

$$P(Z_t < a,\ M_t^Z \geq b) = \int_{\infty}^{b} \int_{a}^{\infty} f_{Z_t,\ M_t^Z}(x,y) dx dy\ ,$$

Therefore

$$\int_b^{\infty} \int_{-\infty}^{a} f_{Z_t,\ M_t^Z}(x,y) dx dy = \int_{2b-a}^{\infty} \frac{1}{\sigma_{Z_t}\sqrt{2\pi}} \exp\left(-\frac{z^2}{2\sigma_{Z_t}}\right) dz.$$

Thus Calculating the partial differential of both sides in this equation for the variable b,

$$\int_{-\infty}^{a} f_{Z_t,\ M_t^Z}(x,b) dx = \frac{2}{\sigma_{Z_t}\sqrt{2\pi}} \exp\left(-\frac{(2b-a)^2}{2\sigma_{Z_t}}\right).$$

and Calculating the partial differential of both sides in this equation for the variable a, the result of this lemma is derived. i.e.

$$f_{Z_t,\ M_t^Z}(a,b) = \frac{2(2b-a)}{\sigma_{Z_t}^3\sqrt{2\pi}} \exp\left(-\frac{(2b-a)^2}{2\sigma_{Z_t}}\right).$$

□

LEMMA 3.4.3. Assume that

$$dX_t = \sigma(t)dW_t - \frac{1}{2}\sigma(t)^2dt,$$

$$M_t^X = \max{}_{0\leq s\leq t} X_t.$$

Then the joint density function of (X_t, M_t^X) on $\{(a,b) : a \leq b,\ b \geq 0\}$

$$\begin{aligned} & f_{X_t,\ M_t^X}(a,b) \\ = \ & \frac{2(2b-a)}{\sigma_{X_t}^3\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{a-2b+\frac{1}{2}\sigma_{X_t}^2}{\sigma_{X_t}}\right)^2 - y\right) \end{aligned}$$

Proof. Define $\widetilde{W_t}$ such that

$$d\widetilde{W_t} = -\frac{1}{2}\sigma(t)dt + dW_t.$$

And the stochastic process L_t is satisfying the following stochastic differential equation

$$dL_t = \frac{1}{2}\sigma(t)L_t dW_t.$$

Then, L_t is defined as the following equation.

$$L_t = \exp\left(\int_0^t \frac{1}{2}\sigma(u)dW_u - \frac{1}{8}\int_0^t \sigma(u)^2du\right)$$

And the probability measure Q satisfies the following.

$$dQ = L_t\ d\mathbb{Q}$$

Then by the Girsanov Theorem, $\widetilde{W_t}$ is a Brownian motion and X_t is martingale under the measure Q.

Now

$$
\begin{aligned}
& \mathbb{Q}(X_t \le a,\ M_t^X > b) \\
= \ & E[1_{\{X_t \le a,\ M_t^X > b\}}] \\
= \ & E^Q\left[\frac{1_{\{X_t \le a,\ M_t^X > b\}}}{L_t}\right] \\
= \ & \exp\left(-\frac{1}{8}\int_0^t \sigma(t)^2 du\right) E^Q\left[1_{\{X_t \le a,\ M_t^X > b\}} \exp\left(-\frac{1}{2}\int_0^t \sigma(t) dW_u + \frac{1}{4}\int_0^t \sigma(t)^2 du\right)\right] \\
= \ & \exp\left(-\frac{1}{8}\int_0^t \sigma(t)^2 du\right) E^Q\left[1_{\{X_t \le a,\ M_t^X > b\}} e^{-\frac{1}{2}X_t}\right] \\
= \ & \exp\left(-\frac{1}{8}\int_0^t \sigma(t)^2 du\right) \int_b^\infty \int_{-\infty}^a e^{\frac{1}{2}x} f_{X_t,\ M_t^X}(x,y) dx dy \\
= \ & \exp\left(-\frac{1}{8}\sigma_{X_t}^2\right) \int_{-\infty}^a \int_b^\infty e^{-\frac{1}{2}x} \cdot \frac{2(2y-x)}{\sigma_{X_t}^3\sqrt{2\pi}} \exp\left(-\frac{(2y-x)^2}{2\sigma_{X_t}^2}\right) dy dx \\
= \ & \exp\left(-\frac{1}{8}\sigma_{X_t}^2\right) \cdot \frac{1}{\sigma_{X_t}\sqrt{2\pi}} \int_{-\infty}^a \exp\left(-\frac{1}{2}x\right) \exp\left(-\frac{(2b-x)^2}{2\sigma_{X_t}^2}\right) dx \\
= \ & -\frac{b}{\sigma_{X_t}\sqrt{2\pi}} \int_{-\infty}^a \exp\left(-\frac{\{x + \frac{1}{2}(\sigma_{X_t}^2 - 4b))\}^2}{2\sigma_{X_t}^2}\right) dx \\
= \ & e^{-b} \int_{-\infty}^{\frac{a-2b+\frac{1}{2}\sigma_{X_t}^2}{\sigma_{X_t}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\
= \ & e^{-b} N\left(\frac{a - 2b + \frac{1}{2}\sigma_{X_t}^2}{\sigma_{X_t}}\right)
\end{aligned}
$$

where

$$\sigma_{X_t}^2 = \mathrm{Var}(X_t) = \int_0^t \sigma(t)^2 du.$$

Thus

$$\int_b^\infty \int_{-\infty}^a f_{X_t,\ M_t^X}(x,y) dx dy = e^{-b} N\left(\frac{a - 2b + \frac{1}{2}\sigma_{X_t}^2}{\sigma_{X_t}}\right).$$

Calculating the partial differential of both sides in this equation for the variable b,

$$
\begin{aligned}
& \int_{-\infty}^{a} f_{X_t,\ M_t^X}(x,b)dx \\
= \quad & e^{-b} N\left(\frac{a-2b+\frac{1}{2}\sigma_{X_t}^2}{\sigma_{X_t}}\right) + e^{-b}\frac{2}{\sigma_{X_t}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{a-2b+\frac{1}{2}\sigma_{X_t}^2}{\sigma_{X_t}}\right)^2\right\}.
\end{aligned}
$$

And calculating the partial differential of both sides in this equation for the variable b,

$$
\begin{aligned}
& f_{X_t,\ M_t^X}(a,b) \\
= \quad & \frac{2(2b-a)}{\sigma_{X_t}^3\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{a-2b+\frac{1}{2}\sigma_{X_t}^2}{\sigma_{X_t}}\right)^2 - y\right\}
\end{aligned}
$$

□

3.4.1 Up-and-Out Call

Let K be the strike price of an up and out call barrier option and B be the barrier of the option. Then the payoff of the option with the maturity S is

$$
\begin{aligned}
\Phi(I_S) &= (I_S - K)^+ 1_{\{M_S^I \leq B\}} \\
&= (I_S - K) 1_{\{I_S \geq K,\ M_S^I \leq B\}} \\
&= (I e^{X_S} - K) 1_{\{X_S \geq k,\ M_S^X \leq b\}}
\end{aligned}
$$

where

$$
k = \log\frac{K}{I},\ b = \log\frac{B}{I}.
$$

First of all, Let us assume that $I(0) = I \leq B$ so that $b > 0$. Otherwise, because the option price is zero, it is not meaningful. If $k < 0$, we will integrate on $\{(x,y) : k \leq x \leq y \leq b\}$. On the other hand, if $k < 0$, we will integrate on $\{(x,y) : k \leq x \leq b,\ 0 \leq y \leq b\}$. Eventually synthesizing both cases, we will calculate the integration on the region $\{(x,y) : k \le x \le y,\ x^+ \le y \le b\}$.

THEOREM 3.4.1 (the pricing formula of up-and-out call options)**.** Suppose that $0 < I(0) < B$ and there are up-and-out call barrier options on the commodity index with the strike price Kk, the maturity S and the barrier B. The underlying commodity index $I(t)$ satisfies the following stochastic differential equation,

$$\begin{cases} dI(u) &= I(u)\sigma(u)dW_u \\ I(0) &= I \end{cases}.$$

Then the price of the up-and-out call options, $C^{UO}(0,I)$ is as follow.

$$\begin{aligned} C^{UO}(0,I) &= D(0,S)\Big[\, I\left(N(d_1(I,K)) - N(d_1(I,B))\right) \\ &\quad -K\left(N(d_2(I,K)) - N(d_2(I,B))\right) \\ &\quad -B\left(N\left(d_1\left(B, \frac{KI}{B}\right)\right) - N(d_1(B,I))\right) \\ &\quad +\frac{KI}{B}\left(N\left(d_2\left(B, \frac{KI}{B}\right)\right) - N(d_2(B,I))\right)\Big] \end{aligned}$$

where

$$d_1(x,y) = \frac{\log\frac{x}{y} + \frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}},$$
$$d_2(x,y) = d_1(x,y) - \sigma_{X_S}$$

and N is the cumulative distribution function of the standard normal distribution.

Proof. By the Theorem 4.1.1, the price of the up-and-out call options $C^{UO}(0,I)$ is

$$C(0,I) = D(0,S)E[\Phi(I_S)].$$

Here,

$$
\begin{aligned}
& E\left[\Phi(I_S)\right] \\
= \;& E\left[(Ie^{X_S}-K)1_{\{X_S\geq k,\ M_S^X\leq b\}}\right] \\
= \;& \int_k^b\int_{x^+}^b (I_S-K)\frac{1}{\sigma_{X_S}\sqrt{2\pi}}\cdot\frac{2(2y-x)}{\sigma_{X_S}^2}\exp\left\{-\frac{1}{2}\left(\frac{x-2y+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2-y\right\}dydx \\
= \;& \int_k^b\int_{x^+}^b (Ie^x-K)\frac{1}{\sigma_{X_S}\sqrt{2\pi}}\cdot\frac{2(2y-x)}{\sigma_{X_S}^2}\exp\left\{-\frac{1}{2}\left(\frac{x-2y+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2-y\right\}dydx \\
= \;& \int_k^b (Ie^x-K)\frac{1}{\sigma_{X_S}\sqrt{2\pi}}\left[\exp\left\{-\frac{1}{2}\left(\frac{x-2y+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2-y\right\}\right]_{y=b}^{x^+}dx \\
= \;& \int_k^b (Ie^x-K)\frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2-x\right\}dx \\
& -\int_k^b (Ie^x-K)\frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-2b+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2-b\right\}dx
\end{aligned}
$$

$$
\begin{aligned}
= \;& I\int_k^b e^x\frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2-x\right\}dx \\
& -K\int_k^b \frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2-x\right\}dx \\
& -I\int_k^b e^x\frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-2b+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2-b\right\}dx \\
& +K\int_k^b \frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-2b+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2-b\right\}dx.
\end{aligned}
$$

Let us attach the number $\mathrm{I}, \mathrm{II}, \mathrm{III}$ and IV to these four integrations respectively in order and calculate them.

$$
\begin{aligned}
\mathrm{I} &= I\int_k^b \frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2\right\}dx \\
&= I\int_{(b-\frac{1}{2}\sigma_{X_S}^2)/\sigma_{X_S}}^{(k-\frac{1}{2}\sigma_{X_S}^2)/\sigma_{X_S}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx \\
&= I\left\{N\left(\frac{b-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)-N\left(\frac{k-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)\right\} \\
&= I\left\{N\left(\frac{\log\frac{B}{I}-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)-N\left(\frac{\log\frac{K}{I}-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)\right\} \\
&= I\left\{N\left(\frac{\log\frac{I}{K}+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)-N\left(\frac{\log\frac{I}{B}+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{II} &= -K\int_k^b \frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2\right\}dx \\
&= -K\int_{(b+\frac{1}{2}\sigma_{X_S}^2)/\sigma_{X_S}}^{(k+\frac{1}{2}\sigma_{X_S}^2)/\sigma_{X_S}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx \\
&= -K\left\{N\left(\frac{\log\frac{B}{I}+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)-N\left(\frac{\log\frac{K}{I}+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)\right\} \\
&= -K\left\{N\left(\frac{\log\frac{I}{K}-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)-N\left(\frac{\log\frac{I}{B}-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{III} &= -I\int_k^b \frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-2b+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2 + x - b\right\}dx \\
&= -Ie^b\int_k^b \frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-2b+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2 + x\right\}dx \\
&= -B\int_{(-b-\frac{1}{2}\sigma_{X_S}^2)/\sigma_{X_S}}^{(k-2b-\frac{1}{2}\sigma_{X_S}^2)/\sigma_{X_S}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx \\
&= -B\left\{N\left(-\frac{\log\frac{B}{I}+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right) - N\left(-\frac{\log\frac{B^2}{KI}+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)\right\} \\
&= -B\left\{N\left(\frac{\log\frac{B^2}{KI}+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right) - N\left(\frac{\log\frac{B}{I}+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{IV} &= Ke^{-b}\int_k^b \frac{1}{\sigma_{X_S}\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-2b+\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)^2\right\}dx \\
&= K\frac{I}{B}\int_{(-b+\frac{1}{2}\sigma_{X_S}^2)/\sigma_{X_S}}^{(k-2b+\frac{1}{2}\sigma_{X_S}^2)/\sigma_{X_S}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)dx \\
&= \frac{KI}{B}\left\{N\left(-\frac{\log\frac{B}{I}-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right) - N\left(-\frac{\log\frac{B^2}{KI}-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)\right\} \\
&= \frac{KI}{B}\left\{N\left(\frac{\log\frac{B^2}{KI}-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right) - N\left(\frac{\log\frac{B}{I}-\frac{1}{2}\sigma_{X_S}^2}{\sigma_{X_S}}\right)\right\}
\end{aligned}
$$

$$\therefore C^{UO}(0,I) = D(0,S)(\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}).$$

This equation is equal with the conclusion of this theorem. □

REMARK 3.4.1. Let $C(0,I;K)$ be the price of vanilla call option at time zero with the strike price K and $C^{UO}(0,I;K)$ be the price of up-and-out call option at time zero with the strike price K and the barrier B . Then the price $C^{UO}(0,I;K)$ is represented as follow.

$$
\begin{aligned}
C^{UO}(0,I;K) &= C(0,I;K) - C(0,I;B) - C\left(0,B;\frac{KI}{B}\right) \\
&\quad + BN(d_1(B,I)) - \frac{KI}{B}N(d_2(B,I))
\end{aligned}
$$

Corollary 3.4.1 (the pricing formula of up-and-in call options). Let $C^{UI}(0,I;K)$ be the price of up-and-in call option at time zero with the strike price K and barrier B. Then

$$C^{UI}(0,I;K) = C(0,I;B)+C\left(0,B;\frac{KI}{B}\right)-BN(d_1(B,I))+\frac{KI}{B}N(d_2(B,I)).$$

Proof.

$$C^{UI}(0,I;K) = C(0,I;K) - C^{UO}(0,I;K).$$

□

3.5 Greeks

In this chapter, we will calculate the greeks of European options on the commodity index using the pricing formulas in Section 3.2.

3.5.1 Delta

Delta is the sensitivity of option prices for the change of the index values. Delta of European call option is represented as follow.

$$\frac{\partial C(t,I)}{\partial I}.$$

Now let us calculate the delta of European call options.

THEOREM 3.5.1. The delta of European call option is $N(d_1)$.

Proof. By Theorem 3.2.1

$$\frac{\partial C}{\partial I} = N(d_1) + IN'(d_1)\frac{\partial d_1}{\partial I} - KN'(d_2)\frac{\partial d_2}{\partial I}.$$

Define $\sigma := \sigma(t)$ in Assumption 5. Since

$$\frac{\partial d_1}{\partial I} = \frac{\partial d_1}{\partial I} = \frac{1}{I\|\sigma\|_{\mathscr{L}^2}},$$

$$\frac{\partial C}{\partial I} = N(d_1) + \frac{1}{\|\sigma\|_{\mathscr{L}^2}\sqrt{2\pi}}\exp\left(-\frac{d_1^2}{2}\right) - \frac{K}{I}\frac{1}{\|\sigma\|_{\mathscr{L}^2}\sqrt{2\pi}}\exp\left(-\frac{d_2^2}{2}\right)$$

However

$$\begin{aligned} \exp\left(-\frac{d_1^2}{2}\right) &= \exp\left(-\frac{\left(d_2 + \|\sigma\|_{\mathscr{L}^2}\right)^2}{2}\right) \\ &= \exp\left(-\frac{d_2^2}{2}\right)\exp\left(-d_2\|\sigma\|_{\mathscr{L}^2} + \frac{\|\sigma\|_{\mathscr{L}^2}^2}{2}\right) \\ &= \exp\left(-\frac{d_2^2}{2}\right)\exp\left(\frac{K}{I}\right). \end{aligned}$$

Therefore

$$\frac{\partial C(t, I)}{\partial I} = N(d_1).$$

□

This formula is used in the hedging portfolio of our commodity index which appear in the process inducing PDEs in Section 3.1.

3.5.2 Vega

Volatilities of commodity futures prices have a term structure as the functions for expiry of futures contract. And according to Taleb(1997)[11], instantaneous forward volatility of some derivatives at specific time is represented by the implied volatility of the derivatives matured at same time, if the volatility term structure is differentiable. i.e.

$$\sigma_t^2 = \sigma(0,t)^2$$

where σ_t is the forward volatility at time t and $\sigma_{0,t}$ is the implied volatility of the derivative matures at time t. Then

$$\int_0^S \sigma(t)^2 dt = \int_0^S \sigma(0,s)^2 ds$$

Here $\sigma(0,s)$ means the volatility term structure at present. Therefore, d_1 and d_2 in the values of call options on commodity index are represented as

$$d_1 = \frac{\log \frac{I}{K} + \frac{1}{2}\int_t^S \sigma(0,s)^2 ds}{\sqrt{\int_t^S \sigma(0,s)^2 ds}} \tag{3.5.1}$$

and

$$d_2 = d_1 - \sqrt{\int_t^S \sigma(0,s)^2 ds}. \tag{3.5.2}$$

So Vega, the sensitivity for volatilities of commodity futures prices, has to been represented in concept of the Frechet derivatives for the volatility term structure.

We will consider Vega at present time 0 in this subsection. And we will abbreviate the volatility term structure $\sigma(0,s)$ to $\sigma(s)$. Then d_1 and d_2 are represented in this subsection as

$$d_1 = \frac{\log \frac{I}{K} + \frac{1}{2}\int_0^S \sigma(s)^2 ds}{\sqrt{\int_0^S \sigma(s)^2 ds}} \tag{3.5.3}$$

and

$$d_2 = d_1 - \sqrt{\int_0^S \sigma(s)^2 ds}. \tag{3.5.4}$$

Namely $\sigma(s)$ of the equations (3.5.3) and (3.5.4) is the volatility term structure in this subsection.

Definition 3.5.1. Assume that $f, g \in \mathscr{L}^2[0,S]$. The inner product of f and g in $\mathscr{L}^2[0,S]$ is as follow.

$$f \circ g = \int_0^S fg\ ds.$$

Definition 3.5.2. Let $V(t, I, K, S, \sigma, r)$ be the value of an European call option on the commodity index. The function F is defined as follow.

$$\begin{aligned} F : \mathscr{L}^2[0,S] &\longrightarrow \mathbb{R} \\ \sigma &\longmapsto V(t, I, K, S, \sigma, r), \end{aligned}$$

for arbitrary but fixed $t \in [0,S]$. Here K is the strike price and S is the maturity of the option.

We want to define the Frechet defivative $DF(\sigma)$ of F at σ such that

$$\lim_{h \to 0} \frac{|F(\sigma + h) - F(\sigma) - DF(\sigma)(h)|}{\|h\|_{\mathscr{L}^2[0,S]}} = 0$$

where $\sigma,\ h \in \mathscr{L}^2[0,S]$. First we need next lemma to get the linear functional $DF(\sigma)$.

LEMMA 3.5.1. (limit of Frechet derivatives sequance) Consider sequences of functions $\{\sigma_n\}, \{h_n\} \subset \mathscr{L}^2[0,S]$, where $\sigma_n \to \sigma,\ h_n \to h$ and $\sigma,\ h \in \mathscr{L}^2[0,S]$. If Ψ is continuously differentiable at σ in $\mathscr{L}^2[0,S]$, then

$$D\Psi(\sigma_n)(h_n) \to D\Psi(\sigma)(h).$$

Proof. Since Ψ is continuously differentiable

$$D\Psi(\sigma) : \mathscr{L}^2[0,S] \to \mathbb{R}$$

and

$$D\Psi : \mathscr{L}^2[0,S] \to (\mathscr{L}^2[0,S])^*$$

are continuous and bounded. Here $(\mathscr{L}^2[0,S])^*$ is the dual space of $\mathscr{L}^2[0,S]$. Then

$$
\begin{aligned}
& |D\Psi(\sigma_n)(h_n) - D\Psi(\sigma)(h)| \\
\leq \; & |D\Psi(\sigma_n)(h_n) - D\Psi(\sigma_n)(h)| + |D\Psi(\sigma_n)(h) - D\Psi(\sigma)(h)| \\
\leq \; & \|D\Psi(\sigma_n)\|_{(\mathscr{L}^2[0,S])^*}\|h_n - h\|_{\mathscr{L}^2[0,S]} + \|D\Psi(\sigma_n) - D\Psi(\sigma)\|_{(\mathscr{L}^2[0,S])^*}\|h\|_{\mathscr{L}^2[0,S]}
\end{aligned}
$$

where

$$\|\mathscr{A}\|_{(\mathscr{L}^2[0,S])^*} = \sup\left\{|\mathscr{A}(f)| : f \in \mathscr{L}^2[0,S],\ \|f\|_{(\mathscr{L}^2[0,S])^*} \leq 1\right\}$$

for some linear bounded operator $\mathscr{A}$. $\|D\Psi(\sigma_n)\|_{(\mathscr{L}^2[0,S])^*}$ and $\|h\|_{\mathscr{L}^2[0,S]}$ are bounded. In addition $\|h_n - h\|_{\mathscr{L}^2[0,S]} \to 0$ and $\|D\Psi(\sigma_n) - D\Psi(\sigma)\|_{(\mathscr{L}^2[0,S])^*} \to 0$ as $n \to \infty$. Therefore the conclusion of this lemma holds. $\square$

Definition 3.5.3. $\{u_j(s) : [0,S] \to \mathbb{R} \mid j = 0,1,2...\}$ is an arbitrary countable orthonormal basis in the function space $\mathscr{L}^2[0,S]$.

And $\bar{\sigma}_n$ means that the projection of σ to the subspace $< u_0, ..., u_n >$ of $\mathscr{L}^2[0,S]$ which is the space generated by $\{u_0, ..., u_n\}$ in $\mathscr{L}^2[0,S]$.

REMARK 3.5.1. By Definition 3.5.3. σ and $\bar{\sigma}_n$ is represented as follows.

$$\sigma(s) = \Sigma_{j=0}^{\infty} a_j u_j(s),$$

$$\bar{\sigma}_n(s) = \Sigma_{j=0}^{n} a_j u_j(s)$$

for $s \in [0,S]$ and $a_j \in \mathbb{R}$. Then next equality holds.

$$\lim_{n\to\infty} \bar{\sigma}_n(s) = \sigma(s).$$

THEOREM 3.5.2. (the Frechet derivatives of European call options for volatilities) Suppose that U is bounded open subset in $\mathscr{L}^2[0,S]$, $\sigma \in U$, $\|\sigma\|_{\mathscr{L}^2[0,S]} \neq 0$ and F is continuously differentiable in $\mathscr{L}[0,S]$.

Furthermore $r(s) \in \mathscr{L}^2[0,S]$ such that $\exp\left(-\int_0^S r(s)ds\right) = D(0,S)$.

Then $DF(\sigma)$ satisfying the equation

$$\lim_{h\to 0} \frac{|F(\sigma+h) - F(\sigma) - DF(\sigma)(h)|}{\|h\|_{\mathscr{L}^2[0,S]}} = 0$$

for any $h \in \mathscr{L}^2[0,S]$ is

$$D(0,S)[IN'(d_1(\sigma))\delta_1(\sigma) - KN'(d_2(\sigma))\delta_2(\sigma)](\sigma \circ h)$$

where

$$\delta_1 = \frac{-\log\frac{I}{K} + \frac{1}{2}\int_0^S \sigma(s)^2 ds}{\left(\int_t^S \sigma(s)^2 ds\right)^{3/2}}$$

and

$$\delta_2 = \frac{-\log\frac{I}{K} - \frac{1}{2}\int_0^S \sigma(s)^2 ds}{\left(\int_t^S \sigma(s)^2 ds\right)^{3/2}}.$$

Proof. Let us define $\bar{h}_n$ as the projection of h on $< u_0, \cdots, u_n >$ for some $h \in \mathscr{L}^2[0,S]$, that is, $\bar{h}_n = \Sigma_{j=0}^n h_j u_j$.

We can consider the least integer $N > 0$ such that $\bar{\sigma}_n \neq 0$ for all $n \geq N$. Then for $n \geq N$

$$0 < \|\bar{\sigma}_N\|^2_{\mathscr{L}^2[0,S]} \leq \|\bar{\sigma}_n\|^2_{\mathscr{L}^2[0,S]} \leq \|\sigma\|^2_{\mathscr{L}^2[0,S]} < \infty. \tag{3.5.5}$$

First, we will calculate the Gâteaux derivative of F at $\bar{\sigma}_n$. Define

$$F(\bar{\sigma}_n) = F(a_0u_0 + \cdots + a_nu_n) =: \tilde{F}(a_0, a_1, ..., a_n).$$

In other word, the function $\tilde{F}$ is defined on $(n+1)$-dimensional vector space.

Then the Gâteaux derivative of F at $\bar{\sigma}_n$ to direction $\bar{h}_n$ is

$$\begin{aligned}
& \lim_{\tau\to 0}\frac{F(\bar{\sigma}_n+\tau\bar{h}_n)-F(\bar{\sigma}_n)}{\tau} \\
= & \lim_{\tau\to 0}\frac{\tilde{F}(a_0+\tau h_1,...,a_n+\tau h_n)-\tilde{F}(a_0,...,a_n)}{\tau} \\
= & \frac{d}{d\tau}\tilde{F}(a_0+\tau h_1,...,a_n+\tau h_n)\,|_{\tau=0} \\
= & \nabla\tilde{F}\,(a_0,...,a_n)\circ(h_0,...,h_n)\,.
\end{aligned}$$

Because

$$\begin{aligned}
\tilde{F}(a_0,a_1,...,a_n) &= F(\bar{\sigma}_n) \\
&= D(0,S)\left[IN(d_1(\bar{\sigma}_n))-KN(d_2(\bar{\sigma}_n))\right]
\end{aligned}$$

where

$$d_1(\bar{\sigma}_n)=\frac{\log\frac{I}{K}+\frac{1}{2}\int_0^S\bar{\sigma}_n^2ds}{\sqrt{\int_0^S\bar{\sigma}_n^2ds}}$$

$$d_2(\bar{\sigma}_n)=\frac{\log\frac{I}{K}-\frac{1}{2}\int_0^S\bar{\sigma}_n^2ds}{\sqrt{\int_0^S\bar{\sigma}_n^2ds}},$$

therefore,

$$\frac{\partial\tilde{F}(a_0,a_1,...,a_n)}{\partial a_j}=D(0,S)\frac{\partial}{\partial a_j}\left(IN(d_1)-KN(d_2)\right)$$

for $j=0,1,...,n$.

Calculating the partial differentials $\dfrac{\partial d_1}{\partial a_j}$ and $\dfrac{\partial d_2}{\partial a_j}$,

$$\begin{aligned}
\frac{\partial d_1}{\partial a_j} &= \frac{-\log\frac{I}{K}+\frac{1}{2}\int_0^S\bar{\sigma}_n^2ds}{\left(\int_0^S\bar{\sigma}_n^2ds\right)^{3/2}}\int_0^S\left(\Sigma_{j=0}^n a_ju_j(s)\right)u_j(s)ds \\
\frac{\partial d_2}{\partial a_j} &= \frac{-\log\frac{I}{K}-\frac{1}{2}\int_0^S\bar{\sigma}_n^2ds}{\left(\int_0^S\bar{\sigma}_n^2ds\right)^{3/2}}\int_0^S\left(\Sigma_{j=0}^n a_ju_j(s)\right)u_j(s)ds.
\end{aligned}$$

Let

$$\delta_1(\bar{\sigma}_n) = \frac{-\log\frac{I}{K} + \frac{1}{2}\int_0^S \bar{\sigma}_n^2 ds}{\left(\int_0^S \bar{\sigma}_n^2 ds\right)^{3/2}}$$

and

$$\delta_2(\bar{\sigma}_n) = \frac{-\log\frac{I}{K} - \frac{1}{2}\int_0^S \bar{\sigma}_n^2 ds}{\left(\int_0^S \bar{\sigma}_n^2 ds\right)^{3/2}},$$

then

$$\frac{\partial \tilde{F}}{\partial a_j} = D(0,S)\left[IN'\left(d_1(\bar{\sigma}_n)\right)\delta_1(\bar{\sigma}_n) - KN'\left(d_2(\bar{\sigma}_n)\right)\delta_2(\bar{\sigma}_n)\right]\int_0^S (\Sigma a_j u_j(s))u_j(s)ds.$$

Therefore

$$\begin{aligned}
& \lim_{\tau\to 0}\frac{F(\bar{\sigma}_n + \tau\bar{h}_n) - F(\bar{\sigma}_n)}{\tau} \\
= \;& \nabla\tilde{F}(a_0,...,a_n)\circ(h_0,...,h_n) \\
= \;& D(0,S)\left[IN'(d_1(\bar{\sigma}_n))\delta_1(\bar{\sigma}_n) - KN'(d_2(\bar{\sigma}_n))\delta_2(\bar{\sigma}_n)\right](\bar{\sigma}_n\circ\bar{h}_n).
\end{aligned}$$

However

$$\begin{aligned}
& |D(0,S)\left(IN'(d_1(\bar{\sigma}_n))\delta_1(\bar{\sigma}_n) - KN'(d_2(\bar{\sigma}_n))\delta_2(\bar{\sigma}_n)\right)| \\
\leq \;& |D(0,S)|\left(|(IN'(d_1(\bar{\sigma}_n))\delta_1(\bar{\sigma}_n)| + |KN'(d_2(\bar{\sigma}_n))\delta_2(\bar{\sigma}_n)|\right).
\end{aligned}$$

Here, by (4.2.1)

$$d_1(\bar{\sigma}_n) = \frac{\log\frac{I}{K}}{\|\bar{\sigma}_n\|_{\mathscr{L}^2[0,S]}} + \frac{1}{2}\|\bar{\sigma}_n\|_{\mathscr{L}^2[0,S]}$$

and

$$|\delta_1(\bar{\sigma}_n)| = \left|\frac{-\log\frac{I}{K}}{\|\bar{\sigma}_n\|^3_{\mathscr{L}^2[0,S]}} + \frac{1}{2\|\bar{\sigma}_n\|_{\mathscr{L}^2[0,S]}}\right|$$

are bounded for $n \geq N$, therefore

$$|(IN'(d_1(\bar{\sigma}_n))\delta_1(\bar{\sigma}_n)|$$

is bounded. Similarly

$$|KN'(d_2(\bar{\sigma}_n))\delta_2(\bar{\sigma}_n)|$$

is bounded. So

$$|D(0,S)\left(IN'(d_1(\bar{\sigma}_n))\delta_1(\bar{\sigma}_n) - KN'(d_2(\bar{\sigma}_n))\delta_2(\bar{\sigma}_n)\right)|$$

is bounded for $n \geq N$.

Let us define the functional $DF(\bar{\sigma}) : U' \to \mathbb{R}$ as

$$DF(\bar{\sigma})(g) = D(0,S)[IN'(d_1(\bar{\sigma}_n))\delta_1(\bar{\sigma}_n) - KN'(d_2(\bar{\sigma}_n))\delta_2(\bar{\sigma}_n)](\bar{\sigma}_n \circ g)$$

for some $g \in U'$ where U' is a bounded open subset of $\mathscr{L}^2[0,S]$.

Then the functional $DF(\bar{\sigma}_n)$ satisfies the continuity and the linearity for g, and boundedness of $DF(\bar{\sigma}_n)$ holds. Therefore, the functional $DF(\bar{\sigma}_n)$ is the Frechet derivatives of F at $\bar{\sigma}_n$.

By Lemma 3.5.1

$$DF(\bar{\sigma}_n)(\bar{h}_n) \to DF(\sigma)(h)$$

as $n \to \infty$.

$\square$

Chapter 4

The Commodity Index Discretely Rebalanced

In chapter 2, we constructed the commodity index rebalanced continuously as soon as the futures prices change. In this chapter, we will consider the commodity index rebalanced discretely. In other words, the rebalancing coefficients are changed from time to time. The following model is based on the structure of RICI.

4.1 Modeling for The Commodity Index Discretely Rebalanced

Assumption 6. Here, we will see the structure of RICI.

- $X_i(t,T)$ $(i = 1, 2, ..., N)$ are the futures prices consisting the commodity index for some T. These futures prices have deterministic volatili-

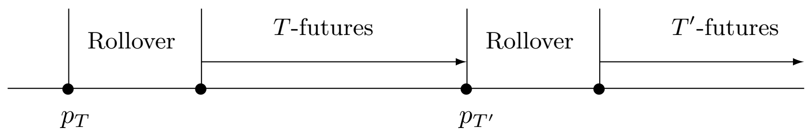

Figure 4.1: the commodity index rebalanced discontinuously

ties $\sigma_i(t,T)$ which is $\mathbb{R}^d$-functions and satisfy the following SDE.

$$dX_i = X_i\sigma_i(t,T)\cdot d\mathbf{W}_t.$$

And $\mathbf{W}_t$ is d-dimensional vector with d independent wiener processes as its elements.

- w_i are the index weight of $X_i(t,T)$.

- $m_i(T)$ are the contract weights determined at the initial time of a roll period in which the underlying futures contracts of commodity index are rolled to $X_i(t,T)$ for each i. These value are only valid to the futures contracts with the delivery month T. $m_i(T)$'s are determined as the following equations.

$$\begin{aligned}\frac{m_i(T)X_i(p_T,T)}{\Sigma_{k=1}^N\, m_k(T)X_k(p_T,T)} &= w_i \quad \text{for all} \quad i,\\ \Sigma_{k=1}^N\, m_k(T) &= 1\end{aligned}$$

  where p_T is the initial time of the roll period in which the underlying futures prices of commodity index are rolled to $X_i(t,T)$.

- The excess return of commodity index is defined as follow during non roll periods.

$$\frac{I(t+\Delta t)}{I(t)} = \frac{\Sigma_{k=1}^N\, m_k(T)X_k(t+\Delta t,T)}{\Sigma_{k=1}^N\, m_k(T)X_k(t,T)}.$$

- During roll periods, the excess return of commodity index is defined as follow.

$$\frac{I(t+\Delta t)}{I(t)} = \frac{C\Big(\alpha(t)\Sigma_{k=1}^{N}\ m_k(T)X_k(t+\Delta t,T)\Big)+\Big((1-\alpha(t))\Sigma_{k=1}^{N}\ m_k(T')X_k(t+\Delta t,T')\Big)}{C\Big(\alpha(t)\Sigma_{k=1}^{N}\ m_k(T)X_k(t,T)\Big)+\Big((1-\alpha(t))\Sigma_{k=1}^{N}\ m_k(T')X_k(t,T')\Big)}$$

  where

$$C = \frac{\Sigma_{k=1}^{N}\ m_k(T')X_k(p_{T'},T')}{\Sigma_{k=1}^{N}\ m_k(T)X_k(p_{T'},T')}$$

  and $p_{T'}$ is the initial time of the roll period in which the underlying futures prices are changed from $X_i(t,T)$ to $X_i(t,T')$. C is constant during the roll period. C is used for continuity of the value process of commodity index.

4.1.1 Non Roll Periods

By the Assumption 6, the dynamic of the commodity index $I(t)$ is represented as follow.

$$\begin{aligned}\frac{dI_t}{I_t} &= \frac{\Sigma_{k=1}^{N}\ m_k(T)dX_t}{\Sigma_{k=1}^{N}\ m_k(T)X_k(t,T)} \\ &= \frac{\Sigma_{k=1}^{N}\ m_k(T)X_k(t,T)\vec{\sigma}_k(t,T)}{\Sigma_{k=1}^{N}\ m_k(T)X_k(t,T)} \cdot d\mathbf{W}_t.\end{aligned}$$

Let

$$\nu_i(t, X_1(t,T), \cdots, X_N(t,T)) = \frac{m_i(T)X_i(t,T)}{\Sigma_{k=1}^{N} m_k(T)X_k(t,T)}$$

for all $i = 1, 2, \cdots, N$. Then

$$\Sigma_{k=1}^{N}\ \nu_i(t, X_1(t,T), \cdots, X_N(t,T)) = 1$$

and $\nu_i > 0$ for all i. So, we can consider that ν_i are the weights of somethings. Using these definitions of the weights ν_i, we can get the following definition of the commodity index.

Definition 4.1.1. Let $\sigma_i(t,T)$ be the deterministic $\mathbb{R}^d$-functions for each $i = 1, 2, \cdots, N$. Then the commodity index which is rebalanced periodically satisfies the following SDE during non roll periods.

$$dI_t \;=\; I(t)\sigma(t,T) \cdot d\mathbf{W}_t$$

where

$$\sigma(t,T) = \Sigma_{k=1}^{N}\, \nu_k(t, X_1(t,T), \cdots, X_N(t,T))\sigma_k(t,T)$$

In other words, ν_is are the weights of the volatilities of the futures prices contributing to the volatility of the commodity index. Let us call ν_i the *volatility weights* of the commodity index performing similar role with the index weights w_i in the Subsection 2.3.1. These volatility weights are stochastic process determined by the futures prices $X_i(t,T)$ and the contract weights $m_i(T)$. So, the volatility of the commodity index σ is a stochastic process.

Definition 4.1.2. (the volatility weights ν_is of the commodity index)

$$\nu_i(t, X_1(t,T), \cdots, X_N(t,T)) = \frac{m_i(T)X_i(t,T)}{\Sigma_{k=1}^{N} m_k(T)X_k(t,T)}$$

for all $i = 1, 2, \cdots, N$.

REMARK 4.1.1. When we consider the model for the commodity index which is rebalanced continuously in Chapter 2, if the volatilities of the futures prices forming the commodity index are deterministic, then so is the volatility of the commodity index.

However the commodity index rebalanced discontinuously has the stochastic volatility even though all futures components of the index have deterministic volatilities. Therefore we will deal with stochastic volatility model henceforth.

THEOREM 4.1.1. For each i, the volatility weight ν_i is a geometric Brownian motion represented as follow.

$$d\nu_i(t, X_1, \cdots, X_N) = \nu_i \left(\mu_i(t, X_1, \cdots, X_N)dt + \theta_i(t, X_1, \cdots, X_N) \cdot d\mathbf{W}_t\right)$$

where

$$\begin{aligned} \mu_i(t, X_1, \cdots, X_N) &= \Sigma_{k\neq i}\ \nu_k^2 \|\sigma_k\|^2 - \nu_i(1-\nu_i)\|\sigma_i\|^2 \\ &\quad +2\Sigma_{k,l\ k\neq l}\ \nu_k\nu_l\|\sigma_k\|\|\sigma_l\|\rho_{kl} - \Sigma_{k\neq i}\ \nu_k\|\sigma_k\|\|\sigma_i\|\rho_{ik}, \end{aligned}$$

$$\theta_i(t, X_1, \cdots, X_N) = (1-\nu_i)\sigma_i - \Sigma_{k\neq i}\ \nu_k\sigma_k,$$

and ρ_{kl} is the correlation coefficient of $X_k(t, T)$ and $X_l(t, T)$.

Proof. We will use the definition of ν_i and the Itô formula to prove this theorem. For the sake of convenience, we will write as following notations,

$$m_i = m_i(T),\ X_i = X_i(t, T)$$

and

$$\mathbb{X} = \Sigma_{k=1}^N\ m_i X_i.$$

First, let us calculate the partial derivatives for ν_i.

$$\begin{aligned} \frac{\partial \nu_i}{\partial X_i} &= \frac{m_i\mathbb{X} - m_i^2 X_i}{\mathbb{X}^2} \\ &= \frac{m_i \Sigma_{k\neq i}\ m_k X_k}{\mathbb{X}^2}. \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \nu_i}{\partial X_i^2} &= \frac{\partial}{\partial X_i}\left(\frac{m_i \Sigma_{k\neq i}\ m_k X_k}{\mathbb{X}^2}\right) \\ &= -\frac{2m_i^2 \Sigma_{k\neq i}\ m_k X_k}{\mathbb{X}^3}. \end{aligned}$$

When $k \neq i$,

$$\frac{\partial \nu_i}{\partial X_k} = \frac{-m_i m_k X_i}{\mathbb{X}^2},$$

$$\begin{aligned}\frac{\partial^2 \nu_i}{\partial X_k^2} &= \frac{\partial}{\partial X_k}\left(\frac{-m_i m_k X_i}{\mathbb{X}^2}\right)\\ &= \frac{2m_i m_k^2 X_i}{\mathbb{X}^3},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \nu_i}{\partial X_i \partial X_k} &= \frac{\partial}{\partial X_i}\left(\frac{-m_i m_k X_i}{\mathbb{X}^2}\right)\\ &= \frac{2m_i^2 m_k X_i - m_i m_k \mathbb{X}}{\mathbb{X}^3}.\end{aligned}$$

When $k \neq l$ and $l \neq i$,

$$\begin{aligned}\frac{\partial^2 \nu_i}{\partial X_l \partial X_k} &= \frac{\partial}{\partial X_l}\left(\frac{-m_i m_k X_i}{\mathbb{X}^2}\right)\\ &= \frac{2m_i m_k m_l X_i}{\mathbb{X}^3}.\end{aligned}$$

Thus by the Itô formula

$$\begin{aligned}d\nu_i &= \Sigma_{k=1}^N \frac{\partial \nu_i}{\partial X_k} dX_k + \frac{1}{2}\Sigma_{k=1}^N \frac{\partial^2 \nu_i}{\partial X_k^2}(dX_k)^2 + \Sigma_{k,l\ k\neq l} \frac{\partial^2 \nu_i}{\partial X_k \partial X_l}(dX_k)(dX_l)\\
&= \frac{m_i X_i \Sigma_{k\neq i}\ m_k X_k}{\mathbb{X}^2}\sigma_i \cdot d\mathbf{W}_t - \Sigma_{k\neq i}\frac{m_i X_i m_k X_k}{\mathbb{X}^2}\sigma_k \cdot d\mathbf{W}_t\\
&\quad -\frac{m_i^2 X_i^2 \Sigma_{k\neq i} m_k X_k}{\mathbb{X}^3}\|\sigma_i\|^2 dt + \frac{m_i X_i \Sigma_{k\neq i} m_k^2 X_k^2 \|\vec{\sigma}_k\|^2}{\mathbb{X}^3} dt\\
&\quad +\Sigma_{k\neq i}\left(\frac{2m_i^2 X_i^2 m_k X_k}{\mathbb{X}^3} - \frac{m_i X_i m_k X_k}{\mathbb{X}^2}\right)\|\sigma_i\|\|\sigma_k\|\rho_{ik} dt\\
&\quad +\Sigma_{k,l\ k\neq l,\ l\neq i}\frac{2m_l X_l m_k X_k m_i X_i}{\mathbb{X}^3}\|\sigma_l\|\|\sigma_k\|\rho_{kl} dt\\
&= \nu_i[(1-\nu_i)\sigma_i - \Sigma_{k\neq i}\ \nu_k \sigma_k]\cdot d\mathbf{W}_t\\
&\quad +\nu_i[-\nu_i(1-\nu_i)\|\sigma_i\|^2 + \Sigma_{k\neq i}\ \nu_k^2\|\sigma_k\|^2 + 2\Sigma_{k,l\ k\neq l}\ \nu_k \nu_l \|\sigma_k\|\|\sigma_l\|\rho_{kl}\\
&\quad -\Sigma_{k\neq i}\ \nu_k \|\sigma_k\|\|\sigma_i\|\rho_{ik}] dt.\end{aligned}$$

Therefore ν_i is geometric Brownian motions for each i. $\square$

4.1.2 Roll Periods

By Assumption 6, during roll periods, the value of commodity index is determined as follow.

$$\frac{I(t+\Delta t)}{I(t)} = \frac{C\alpha(t)\Sigma_{k=1}^{N}\ m_k(T)X_k(t+\Delta t,T) + (1-\alpha(t))\,\Sigma_{k=1}^{N}\ m_k(T')X_k(t+\Delta t,T')}{C\alpha(t)\Sigma_{k=1}^{N}\ m_k(T)X_k(t,T) + (1-\alpha(t))\,\Sigma_{k=1}^{N}\ m_k(T')X_k(t,T')}$$

where

$$C = \frac{\Sigma_{k=1}^{N}\ m_k(T')X_k(p_{T'},T')}{\Sigma_{k=1}^{N}\ m_k(T)X_k(p_{T'},T')},$$

$$\alpha(t) = \frac{p_1 - t}{p_1 - p_{T'}}.$$

and $p_{T'}$ is the initial time and p_1 is the terminal time of the roll period in which the futures contracts are rolled from $X_i(t,T)$ to $X_i(t,T')$.

Then the dynamic of the commodity index during the rollover period is as follow.

$$\frac{dI_t}{I_t} = \frac{C\alpha(t)\Sigma_{k=1}^{N}\ m_k(T)dX_k(t,T) + (1-\alpha(t))\,\Sigma_{k=1}^{N}\ m_k(T')dX_k(t,T')}{C\alpha(t)\Sigma_{k=1}^{N}\ m_k(T)X_k(t,T) + (1-\alpha(t))\,\Sigma_{k=1}^{N}\ m_k(T')X_k(t,T')}.$$

Let us define ν_k and ν'_k such that

$$\begin{aligned}
\nu_i &= \nu_i(t, X_1(t,T), ..., X_N(t,T), X_1(t,T'), ..., X_N(t,T')) \\
&= \frac{C\alpha(t)m_i(T)X_i(t,T)}{\mathbb{X}} \\
\nu'_i &= \nu'_i(t, X_1(t,T), ..., X_N(t,T), X_1(t,T'), ..., X_N(t,T')) \\
&= \frac{(1-\alpha(t))m_i(T')X_i(t,T')}{\mathbb{X}}
\end{aligned}$$

for each $i = 1, 2, \cdots, N$ where

$$\mathbb{X} = C\alpha(t)\Sigma_{k=1}^{N}\ m_k(T)X_k(t,T) + (1-\alpha(t))\,\Sigma_{k=1}^{N}\ m_k(T')X_k(t,T').$$

Then $0 < \nu_i < 1,\ 0 < \nu_i' < 1$ for all i and

$$\Sigma_{k=1}^{N}\nu_k + \Sigma_{k=1}^{N}\nu_k' = 1.$$

Assuming

$$\begin{aligned} dX_i(t,T) &= X_i(t,T)\sigma_i(t,T)\cdot d\mathbf{W}_t \\ dX_i(t,T') &= X_i(t,T')\sigma_i(t,T')\cdot d\mathbf{W}_t \end{aligned}$$

we can rewrite the dynamic of the commodity index as follow.

$$\frac{dI_t}{I_t} = \Sigma_{k=1}^{N}\ \nu_k\sigma_k(t,T)\cdot d\mathbf{W}_t + \Sigma_{k=1}^{N}\ \nu_k'\sigma_k(t,T')\cdot d\mathbf{W}_t.$$

Definition 4.1.3. Let $\sigma_i(t,T)$ and $\sigma_i(t,T')$ be the deterministic $\mathbb{R}^d$-functions for each $i = 1, 2, \cdots, N$. Then the commodity index which is rebalanced periodically satisfies the following SDE under risk neutral measure during rollover periods.

$$dI_t = I(t)\sigma(t,T)\cdot d\mathbf{W}_t$$

where

$$\sigma(t,T) = \Sigma_{k=1}^{N}\ \nu_k\sigma_k(t,T) + \Sigma_{k=1}^{N}\ \nu_k'\sigma_k(t,T').$$

Namely, ν_i and ν_i' are volatility weights for the volatility of each futures price composing the commodity index during a rollover period.

Definition 4.1.4. (the volatility weights ν_is and ν_i's of the commodity index during roll periods)

$$\begin{aligned}
\nu_i(t, X_1(t,T), \cdots, X_N(t,T), X_1(t,T'), ..., X_N(t,T')) &= \frac{C\alpha(t)m_i(T)X_i(t,T)}{\mathbb{X}} \\
\nu_i'(t, X_1(t,T), \cdots, X_N(t,T), X_1(t,T'), ..., X_N(t,T')) &= \frac{(1-\alpha(t))m_i(T')X_i(t,T')}{\mathbb{X}}
\end{aligned}$$

for each $i = 1, 2, \cdots, N$.

The proof of next theorem is analogous to the proof of Theorem 5.1.1.

THEOREM 4.1.2. The volatility weights ν_i and ν_i' are a geometric Brownian motion for all i.

Proof. Referring to the proof of Theorem 4.1.1, we only need to check the partial differential of ν_i and ν_i' for time variable t. For convenience, We use the following abbreviations.

$$\alpha(t) = \alpha_t,\ m_i(t,T) = m_i,\ m_i(t,T') = m_i'$$

$$X_i(t,T) = X_i,\ X_i(t,T') = X_i'.$$

Then

$$\begin{aligned}
\frac{\partial \nu_i}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{C\alpha_t m_i(T)X_i(t,T)}{\mathbb{X}}\right) \\
&= -\frac{1}{3}\frac{Cm_iX_i}{\mathbb{X}} - \frac{C\alpha_t m_i X_i}{\mathbb{X}^2}\left(-\frac{1}{3}C\Sigma_{k=1}^N m_k X_k + \frac{1}{3}\Sigma_{k=1}^N m_k' X_k'\right) \\
&= \frac{1}{\mathbb{X}^2}\left(-\frac{1}{3}C(1-\alpha_t)m_iX_i\Sigma_{k=1}^N m_k'X_k' - \frac{1}{3}C\alpha_t m_i X_i \Sigma_{k=1}^N m_k' X_k'\right) \\
&= -\frac{1}{3}\frac{Cm_iX_i\Sigma_{k=1}^N m_k'X_k'}{\mathbb{X}^2} \\
&= -\frac{1}{3\alpha_t(1-\alpha_t)}\frac{C\alpha_t m_i X_i}{\mathbb{X}}\frac{(1-\alpha_t)\Sigma_{k=1}^N m_k' X_k'}{\mathbb{X}} \\
&= -\frac{1}{3\alpha_t(1-\alpha_t)}\nu_i \Sigma_{k=1}^N \nu_k'
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \nu_i'}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{(1-\alpha_t)m_i'X_i'}{\mathbb{X}}\right) \\
&= \frac{1}{3}\frac{m_i'X_i'}{\mathbb{X}} - \frac{(1-\alpha_t)m_i'X_i'}{\mathbb{X}^2}\left(-\frac{1}{3}C\Sigma_{k=1}^N m_k X_k + \frac{1}{3}\Sigma_{k=1}^N m_k' X_k'\right) \\
&= \frac{1}{\mathbb{X}^2}\left(\frac{1}{3}C\alpha_t m_i'X_i'\Sigma_{k=1}^N m_k X_k + \frac{1}{3}C(1-\alpha_t)m_i'X_i'\Sigma_{k=1}^N m_k X_k\right) \\
&= \frac{1}{3}\,\frac{Cm_i'X_i'\Sigma_{k=1}^N m_k X_k}{\mathbb{X}^2} \\
&= \frac{1}{3\alpha_t(1-\alpha_t)}\,\frac{C\alpha_t\Sigma_{k=1}^N m_k X_k}{\mathbb{X}}\,\frac{(1-\alpha_t)m_i'X_i'}{\mathbb{X}} \\
&= \frac{1}{3\alpha_t(1-\alpha_t)}\nu_i'\Sigma_{k=1}^N \nu_k.
\end{aligned}$$

□

4.2 PDEs for The Options on Commodity Index Rebalanced Discontinuously

The PDEs for the values of the options on the commodity indices rebalanced periodically are not same with the PDEs for the values of the options on the commodity indices rebalanced continuously because the volatilities of the former indices are stochastic processes.

In this chapter, we will derive the PDEs for the values of options on the commodity indices rebalanced periodically. We will consider these problem on non roll periods and on roll periods separately as considered in Section 3.1.

Here, we assume that the commodity index is formed from two futures prices for simpleness of the calculation.

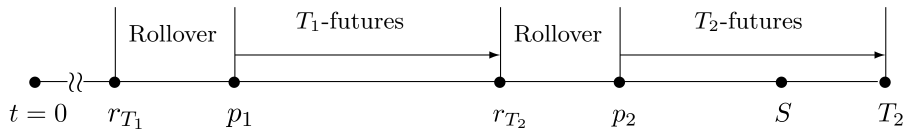

Figure 4.2: underlying futures prices of the commodity index during the time interval $[0, S]$

In Figure 4.2, r_{T_1} and r_{T_2} are the initial times of two rollover periods. And S is the maturity of the option.

4.2.1 Non Roll Periods

First we will derive the PDEs for the option values in the time interval $[p_2, S]$.

Assumption 7.

- We take a portfolio which consists of 1-share European call option on the commodity index and two kinds of futures contracts with the futures prices $X(t, T_2)$ and $Y(t, T_2)$ which are underlying prices of the commodity index. The shares of the futures prices $X(t, T_2)$ and $Y(t, T_2)$ are Δ_X and Δ_Y respectively.

- The dynamic of the futures prices are represented as the followings under risk neutral measure.

$$
\begin{aligned}
dX(t, T_2) &= X(t, T_2)\sigma_X(t, T_2) \cdot d\mathbf{W}_t \\
dY(t, T_2) &= Y(t, T_2)\sigma_Y(t, T_2) \cdot d\mathbf{W}_t.
\end{aligned}
$$

- The dynamic of the commodity index is as follow.

$$dI = I(t)\sigma(t,T_2)\cdot d\mathbf{W}_t$$

  where

$$\sigma(t,T_2) = \nu\sigma_X(t,T_2) + (1-\nu)\,\sigma_Y(t,T_2),$$

$$\begin{aligned}\nu &= \nu(t, X(t,T_2), Y(t,T_2)) \\ &= \frac{m_X(T_2)X(t,T_2)}{m_X(T_2)X(t,T_2) + m_Y(T_2)Y(t,T_2)}.\end{aligned}$$

  $m_X(T_2)$ and $m_Y(T_2)$ which are determined at time r_{T_2} in Figure 4.2 are the contract weights about $X(t,T_2)$ and $Y(t,T_2)$ respectively.

- $\pi(t)$ means the value of this portfolio at time $t \in [p_2, S]$. And $V(t,I,\nu)$ is the value of the European option on the commodity index. Therefore

$$\pi(t) = V(t,I,\nu).$$

By Theorem 4.1.1, the following proposition holds.

COROLLARY 4.2.1. The volatility weight ν satisfies the following SDE.

$$d\nu = \nu\mu(t,\nu)dt + \nu\theta(t,\nu)\cdot d\mathbf{W}_t$$

where

$$\mu(t,\nu) = -(1-\nu)\left(\nu\|\sigma_X\|^2 - (1-\nu)\|\sigma_Y\|^2 - (2\nu-1)\|\sigma_X(t,T_2)\|\|\sigma_Y(t,T_2)\|\rho\right),$$

$$\theta(t,\nu) = (1-\nu)(\sigma_X(t,T_2) - \sigma_Y(t,T_2))$$

and ρ is the correlation coefficient of $X(t,T_2)$ and $Y(t,T_2)$.

Now, let us derive the PDEs for the values of the options during non rollover periods. The change of portfolio value π is represented as follow.

$$d\pi = dV + \Delta_X dX(t,T_2) + \Delta_Y(t,T_2)dY.$$

Because the volatility of the commodity index $\sigma(t,T_2)$ is a function for the stochastic process ν, $d\pi$ is as follow by the Itô formula.

$$\begin{aligned} d\pi &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial I}dI + \frac{\partial V}{\partial \nu}d\nu + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}(dI)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \nu^2}(d\nu)^2 \\ &\quad + \frac{\partial^2 V}{\partial \nu \partial I}(d\nu)(dI) + \Delta_X dX(t,T_2) + \Delta_Y dY(t,T_2). \end{aligned}$$

However by Assumption 7

$$\begin{aligned} dI &= I(t)\left(\nu\sigma_X(t,T_2) + (1-\nu)\sigma_Y(t,T_2)\right)\cdot d\mathbf{W}_t \\ &= I(t)\left(\frac{\nu}{X(t,T_2)}dX + \frac{1-\nu}{Y(t,T_2)}dY\right), \end{aligned}$$

and by Proposition 4.2.1 the diffusion of $d\nu$,

$$\begin{aligned} \nu\theta(t,\nu)\cdot d\mathbf{W}_t &= \nu(1-\nu)(\sigma_X(t,T_2) - \sigma_Y(t,T_2))\cdot d\mathbf{W}_t \\ &= \nu(1-\nu)\left(\frac{dX}{X(t,T_2)} - \frac{dY}{Y(t,T_2)}\right). \end{aligned}$$

Thus

$$\begin{aligned} d\pi &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial I}\ I\left(\frac{\nu}{X(t,T_2)}dX + \frac{1-\nu}{Y(t,T_2)}dY\right) \\ &\quad + \frac{\partial V}{\partial \nu}\ \left(\nu\mu(t,\nu)dt + \nu(1-\nu)\left(\frac{dX}{X(t,T_2)} - \frac{dY}{Y(t,T_2)}\right)\right) \\ &\quad + \left(+\frac{1}{2}\frac{\partial^2 V}{\partial I^2}(I\|\sigma\|)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \nu}(\nu(1-\nu)\|\sigma_X - \sigma_Y\|)^2\right. \\ &\quad \left. + \frac{\partial^2 V}{\partial \nu \partial I}(I\|\sigma\|\nu(1-\nu)\|\sigma_X - \sigma_Y\|\beta)\right)dt \\ &\quad + \Delta_X dX(t,T_2) + \Delta_Y dY(t,T_2) \end{aligned}$$

where β is the correlation coefficient of the index I and the volatility weight w.

REMARK 4.2.1. (*Hedging Portfolio during non roll periods*)

$$\begin{aligned}
\Delta_X &= -\left(\frac{\partial V}{\partial I}\ \frac{\nu I}{X(t,T_2)} + \frac{\partial V}{\partial \nu}\ \frac{\nu(1-\nu)}{X(t,T_2)}\right)\\
\Delta_Y &= -\left(\frac{\partial V}{\partial I}\ \frac{(1-\nu) I}{Y(t,T_2)} - \frac{\partial V}{\partial \nu}\ \frac{\nu(1-\nu)}{Y(t,T_2)}\right),
\end{aligned}$$

Using these hedging portfolio, the diffusion term of $d\pi$ vanishes. Therefore

$$\begin{aligned}
d\pi &= \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \nu}\ \nu\mu(t,\nu) + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}(I\|\sigma\|)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \nu}(\nu(1-\nu)\|\sigma_X - \sigma_Y\|)^2\right.\\
&\quad \left. + \frac{\partial^2 V}{\partial \nu \partial I}(I\|\sigma\|\nu(1-\nu)\|\sigma_X - \sigma_Y\|\beta)\right) dt
\end{aligned}$$

But by the risk neutral arguments

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \nu}\ \nu\mu(t,\nu) + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}(I\|\sigma\|)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \nu}(\nu(1-\nu)\|\sigma_X - \sigma_Y\|)^2 &\\
+ \frac{\partial^2 V}{\partial \nu \partial I}(I\|\sigma\|\nu(1-\nu)\|\sigma_X - \sigma_Y\|\beta) &= r\pi\\
&= rV(t,I,\nu).
\end{aligned}$$

where r is a risk free interest rate. As a result, we can get the following conclusion using the Feynman-Kač Theorems.

THEOREM 4.2.1. Suppose that the process I has the following dynamic

$$\begin{cases} dI(u) = I(u)\sigma(u,T_2)\cdot d\mathbf{W}_u \\ I(t) = I \end{cases}.$$

Also assume that ν is represented as following SDE

$$d\nu = \nu\mu(t,\nu)dt + \nu\theta(t,\nu)\cdot d\mathbf{W}_t$$

and the process

$$\exp\left(-\int_u^S r(\tau)d\tau\right)\left(\frac{dV}{dI}I\sigma + \frac{dV}{d\nu}\nu\theta\right)$$

is in $\mathscr{L}_2$.

If V is a solution to the boundary value problem

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \nu}\ \nu\mu(t,\nu) + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}(I\|\sigma\|)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \nu}\nu^2\|\theta\|^2 & \\ \qquad + \frac{\partial^2 V}{\partial \nu \partial I}(I\|\sigma\|\nu\|\theta\|\beta) = rV(t,I) & , \\ \qquad V(S,I,\nu) = \Phi(I) & \end{cases}$$

where Φ is a S-contingent claim.

Then V has the representation

$$V(t,I,\nu) = \exp\left(-\int_t^S r(u)du\right) E[\Phi(I(S))|\mathscr{F}_t]$$

where $t \in [p_2, S]$ which is the last non rollover period ending at time S.

4.2.2 Roll Periods

Here, we will derive the PDE during the rollover period $[r_{T_2}, p_2]$.

Assumption 8.

- The values of options on the commodity indices are continuous.
- We take a portfolio which consists of 1-share European option on the commodity index and four kinds of futures contracts with the futures prices $X(t,T_1)$, $Y(t,T_1)$, $X(t,T_2)$ and $Y(t,T_2)$. $X(t,T_1)$ and $Y(t,T_1)$ are the first nearby futures. $X(t,T_2)$ and $Y(t,T_2)$ are the second nearby futures. The shares of the futures prices $X(t,T_1)$, $Y(t,T_1)$ $X(t,T_2)$ and $Y(t,T_2)$ are Δ_{X_1}, Δ_{Y_1}, Δ_{X_2} and Δ_{Y_2} respectively.

- The dynamic of the futures prices are as follows.

$$\begin{aligned} dX(t,T_1) &= X(t,T_1)\sigma_X(t,T_1)\cdot d\mathbf{W}_t \\ dY(t,T_1) &= Y(t,T_1)\sigma_Y(t,T_1)\cdot d\mathbf{W}_t \\ dX(t,T_2) &= X(t,T_2)\sigma_X(t,T_2)\cdot d\mathbf{W}_t \\ dY(t,T_2) &= Y(t,T_2)\sigma_Y(t,T_2)\cdot d\mathbf{W}_t \end{aligned}$$

- Let us define C and $\mathbb{X}$ as follows.

$$\begin{aligned} C &= \frac{m_X(T_2)X(r_{T_2},T_2)+m_Y(T_2)Y(r_{T_2},T_2)}{m_X(T_1)X(r_{T_2},T_2)+m_Y(T_1)Y(r_{T_2},T_2)}, \\ \mathbb{X} &= C\alpha(t)\{m_X(T_1)X(t,T_1)+m_Y(T_1)Y(t,T_1)\} \\ &\quad +(1-\alpha(t))\{m_X(T_2)X(t,T_2)+m_X(T_2)Y(t,T_2)\}. \end{aligned}$$

And the volatility weights ν_1, ν_2, ν_3 and ν_4 are defined as

$$\begin{aligned} \nu_1(t,X(t,T_1),Y(t,T_1),X(t,T_2),Y(t,T_2)) &= \frac{C\alpha(t)m_X(T)X(t,T_1)}{\mathbb{X}}, \\ \nu_2(t,X(t,T_1),Y(t,T_1),X(t,T_2),Y(t,T_2)) &= \frac{C\alpha(t)m_Y(T)Y(t,T_1)}{\mathbb{X}}, \\ \nu_3(t,X(t,T_1),Y(t,T_1),X(t,T_2),Y(t,T_2)) &= \frac{(1-\alpha(t))m_X(T')X(t,T_2)}{\mathbb{X}}, \\ \nu_4(t,X(t,T_1),Y(t,T_1),X(t,T_2),Y(t,T_2)) &= \frac{(1-\alpha(t))m_X(T')Y(t,T_2)}{\mathbb{X}} \\ &= 1-(\nu_1+\nu_2+\nu_3) \end{aligned}$$

where $m_X(T_1)$ and $m_Y(T_1)$ are the contract weights determined at time r_{T_1} in Figure 4.2, and $m_X(T_2)$ and $m_X(T_2)$ are the contract weights determined at time r_{T_2} in Figure 4.2.

- The dynamic of the commodity index is as follow.

$$dI = I(t)\sigma(t,T_1,T_2)\cdot d\mathbf{W}_t$$

where

$$\sigma(t,T_1,T_2) = \nu_1\sigma_X(t,T_1) + \nu_2\sigma_Y(t,T_1) + \nu_3\sigma_X(t,T_2) + \nu_4\sigma_Y(t,T_2).$$

- Assume that

$$\begin{aligned}
d\nu_1 &= \nu_1\left(\mu_1(t,\nu_1)dt + \theta_1(t,\nu_1)\cdot d\mathbf{W}_t\right)\\
d\nu_2 &= \nu_2\left(\mu_2(t,\nu_2)dt + \theta_2(t,\nu_2)\cdot d\mathbf{W}_t\right)\\
d\nu_3 &= \nu_3\left(\mu_3(t,\nu_3)dt + \theta_3(t,\nu_3)\cdot d\mathbf{W}_t\right).
\end{aligned}$$

- $\pi(t)$ means the value of this portfolio at time t. And $V(t,I,\nu_1,\nu_2,\nu_3)$ is the value of the European options on the commodity index. Therefore

$$\pi(t) = V(t,I,\nu_1,\nu_2,\nu_3).$$

$V(t,I,\nu_1,\nu_2,\nu_3)$ is the value of the European options on the commodity index during a rollover period. However because $\lim_{t\nearrow p_2}\alpha(t) = 0$,

$$\lim_{t\nearrow p_2}\nu_1 = 0, \quad \lim_{t\nearrow p_2}\nu_2 = 0, \quad \lim_{t\nearrow p_2}\nu_3 = \nu$$

and

$$\nu_1(p_2) = 0, \;\; \nu_2(p_2) = 0,$$

$$\nu_3\left(p_2, X(p_2,T_2), Y(p_2,T_2)\right) = \nu\left(p_2, X(p_2,T_2), Y(p_2,T_2)\right).$$

Therefore we can define the boundary value in this time interval as follows.

$$\lim_{t\nearrow p_2} V(t,I(t),\nu_1,\nu_2,\nu_3) = V(p_2, I_{p_2}, \nu).$$

Let us derive the PDE for the values of the options on the commodity index during rollover periods. The change of the portfolio value π is

$$\begin{aligned}
d\pi &= dV + \Delta_{X_1}dX(t,T_1) + \Delta_{Y_1}dY(t,T_1)\\
&\quad + \Delta_{X_2}dX(t,T_2) + \Delta_{Y_2}dY(t,T_2).
\end{aligned} \tag{4.2.1}$$

Since the volatility of the commodity index $\sigma(t,T)$ is a function for the stochastic processes ν_1, ν_2 and ν_3, $d\pi$ is as follow by the Itô formula.

$$\begin{aligned} d\pi &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial I}dI + \Sigma_{k=1}^3 \frac{\partial V}{\partial \nu_k}d\nu_k + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}(dI)^2 + \frac{1}{2}\Sigma_{k=1}^3 \frac{\partial^2 V}{\partial \nu_k^2}(d\nu_k)^2 \\ &\quad + \Sigma_{k=1}^3 \frac{\partial^2 V}{\partial I \partial \nu_k}(dI)(d\nu_k) + \Sigma_{k\neq l} \frac{\partial^2 V}{\partial \nu_k \partial \nu_l}(d\nu_k)(d\nu_l) \\ &\quad + \Delta_{X_1} dX(t,T_1) + \Delta_{Y_1} dY(t,T_1) + \Delta_{X_2} dX(t,T_2) + \Delta_{Y_2} dY(t,T_2). \end{aligned} \tag{4.2.2}$$

However by Assumption 8

$$\begin{aligned} dI &= I(t)\left(\nu_1\sigma_X(t,T_1) + \nu_2\sigma_Y(t,T_1) + \nu_3\sigma_X(t,T_2) + \nu_4\sigma_Y(t,T_2)\right)\cdot d\mathbf{W}_t \\ &= I(t)\left(\frac{\nu_1}{X(t,T_1)}dX(t,T_1) + \frac{\nu_2}{Y(t,T_1)}dY(t,T_1) + \frac{\nu_3}{X(t,T_2)}dX(t,T_2)\right. \\ &\quad \left. + \frac{\nu_4}{Y(t,T_2)}dY(t,T_2)\right). \end{aligned} \tag{4.2.3}$$

LEMMA 4.2.1. The diffusions of ν_1, ν_2 and ν_3 are respectively

$$\begin{aligned} \nu_1\theta_1(t,\nu_1)\cdot d\mathbf{W}_t &= \nu_1\left\{(1-\nu_1)\frac{dX(t,T_1)}{X(t,T_1)} - \nu_2\frac{dY(t,T_1)}{Y(t,T_1)} - \nu_3\frac{dX(t,T_2)}{X(t,T_2)} - \nu_4\frac{dY(t,T_2)}{Y(t,T_2)}\right\} \\ \nu_2\theta_2(t,\nu_2)\cdot d\mathbf{W}_t &= \nu_2\left\{(1-\nu_2)\frac{dY(t,T_1)}{Y(t,T_1)} - \nu_1\frac{dX(t,T_1)}{X(t,T_1)} - \nu_3\frac{dX(t,T_2)}{X(t,T_2)} - \nu_4\frac{dY(t,T_2)}{Y(t,T_2)}\right\} \\ \nu_3\theta_3(t,\nu_3)\cdot d\mathbf{W}_t &= \nu_3\left\{(1-\nu_3)\frac{dX(t,T_2)}{X(t,T_2)} - \nu_1\frac{dX(t,T_1)}{X(t,T_1)} - \nu_2\frac{dY(t,T_1)}{Y(t,T_1)} - \nu_4\frac{dY(t,T_2)}{Y(t,T_2)}\right\}. \end{aligned}$$

Proof. Suppose a_k are deterministic functions and X_k are distinct stochastic processes of futures prices with the dynamic

$$dX_i = X_i\sigma_i \cdot d\mathbf{W}_t$$

for $1 \leq k \leq 2N$. Let

$$\nu_i = \frac{a_i X_i}{\Sigma_{k=1}^{2N} a_k X_k}$$

where $1 \leq i \leq 2N$. Then

$$\begin{aligned}
\frac{\partial \nu_i}{\partial X_i} &= \frac{a_i}{\Sigma_{k=1}^{2N} a_k X_k} - \frac{a_i X_i}{\left(\Sigma_{k=1}^{2N} a_k X_k\right)^2} \, a_i \\
&= \frac{a_i}{\Sigma_{k=1}^{2N} a_k X_k} \left(1 - \frac{a_i X_i}{\Sigma_{k=1}^{2N} a_k X_k}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial \nu_i}{\partial X_i} dX_i &= \nu_i (1 - \nu_i) \sigma_i \cdot d\mathbf{W}_t \\
&= \nu_i (1 - \nu_i) \frac{dX_i}{X_i}.
\end{aligned}$$

And for $j \neq i$

$$\frac{\partial \nu_i}{\partial X_j} = -\frac{a_i X_i}{\left(\Sigma_{k=1}^{2N} a_k X_k\right)^2} \, a_j.$$

Therefore

$$\begin{aligned}
\frac{\partial \nu_i}{\partial X_j} dX_j &= -\nu_i \nu_j \sigma_j \cdot d\mathbf{W}_t \\
&= -\nu_i \nu_j \frac{dX_j}{X_j}.
\end{aligned}$$

So, we can get the conclusion in this lemma. $\square$

REMARK 4.2.2. (*Hedging Portfolio during roll periods*)

$$\begin{aligned}
\Delta_X &= -\left(\frac{\partial V}{\partial I} \frac{\nu_1 I}{X(t,T_1)} + \frac{\partial V}{\partial \nu_1} \frac{\nu_1(1-\nu_1)}{X(t,T_1)} - \frac{\partial V}{\partial \nu_2} \frac{\nu_1 \nu_2}{X(t,T_1)} - \frac{\partial V}{\partial \nu_3} \frac{\nu_1 \nu_3}{X(t,T_1)}\right) \\
\Delta_Y &= -\left(\frac{\partial V}{\partial I} \frac{\nu_2 I}{Y(t,T_1)} + \frac{\partial V}{\partial \nu_2} \frac{\nu_2(1-\nu_2)}{Y(t,T_1)} - \frac{\partial V}{\partial \nu_1} \frac{\nu_1 \nu_2}{Y(t,T_1)} - \frac{\partial V}{\partial \nu_3} \frac{\nu_2 \nu_3}{Y(t,T_1)}\right) \\
\Delta'_X &= -\left(\frac{\partial V}{\partial I} \frac{\nu_3 I}{X(t,T_2)} + \frac{\partial V}{\partial \nu_3} \frac{\nu_3(1-\nu_3)}{X(t,T_2)} - \frac{\partial V}{\partial \nu_1} \frac{\nu_1 \nu_3}{X(t,T_2)} - \frac{\partial V}{\partial \nu_2} \frac{\nu_2 \nu_3}{X(t,T_2)}\right) \\
\Delta'_Y &= -\left(\frac{\partial V}{\partial I} \frac{\nu_4 I}{Y(t,T_2)} - \frac{\partial V}{\partial \nu_1} \frac{\nu_1 \nu_4}{Y(t,T_2)} - \frac{\partial V}{\partial \nu_2} \frac{\nu_2 \nu_4}{Y(t,T_2)} - \frac{\partial V}{\partial \nu_3} \frac{\nu_3 \nu_4}{Y(t,T_2)}\right).
\end{aligned}$$

Substituting the equation (4.2.3) and these four delta values to the equation (4.2.2), the diffusion term of $d\pi$ vanishes. Therefore

$$\begin{aligned} d\pi &= \frac{\partial V}{\partial t}dt + \Sigma_{k=1}^{3} \frac{\partial V}{\partial \nu_k}\nu_k\mu_k dt + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}(dI)^2 + \frac{1}{2}\Sigma_{k=1}^{3} \frac{\partial^2 V}{\partial \nu_k^2} (d\nu_k)^2 \\ &\quad + \Sigma_{k=1}^{3} \frac{\partial^2 V}{\partial I \partial \nu_k}(dI)(d\nu_k) + \Sigma_{k\neq l} \frac{\partial^2 V}{\partial \nu_k \partial \nu_l}(d\nu_k)(d\nu_l) \qquad (4.2.4) \\ &= \Bigg(\frac{\partial V}{\partial t} + \Sigma_{k=1}^{3} \frac{\partial V}{\partial \nu_k}\nu_k\mu_k + \frac{1}{2}\frac{\partial^2 V}{\partial I^2}(I\|\sigma\|)^2 + \frac{1}{2}\Sigma_{k=1}^{3} \frac{\partial^2 V}{\partial \nu_k^2} (\nu_k\|\theta_k\|)^2 \\ &\quad + \Sigma_{k=1}^{3} \frac{\partial^2 V}{\partial I \partial \nu_k} I\|\sigma\|\nu_k\|\theta_k\|\beta_k + \Sigma_{k\neq l} \frac{\partial^2 V}{\partial \nu_k \partial \nu_l}\nu_k\|\theta_k\|\nu_l\|\theta_l\|\gamma_{kl} \Bigg) dt \end{aligned}$$

$$(4.2.5)$$

where β_k is the correlation coefficient of the commodity index I and the volatility weight ν_k and γ_{kl} is the correlation coefficient of the volatility weights ν_k and ν_l.

Thus by the risk neutral arguments

$$\begin{aligned} &\frac{\partial V}{\partial t} + \Sigma_{k=1}^{3} \frac{\partial V}{\partial \nu_k}\nu_k\mu_k + \frac{1}{2} \frac{\partial^2 V}{\partial I^2}(I\|\sigma\|)^2 + \frac{1}{2}\Sigma_{k=1}^{3} \frac{\partial^2 V}{\partial \nu_k^2} (\nu_k\|\theta_k\|)^2 \\ &+ \Sigma_{k=1}^{3} \frac{\partial^2 V}{\partial I \partial \nu_k} I\|\sigma\|\nu_k\|\theta_k\|\beta_k + \Sigma_{k\neq l} \frac{\partial^2 V}{\partial \nu_k \partial \nu_l}\nu_k\|\theta_k\|\nu_l\|\theta_l\|\gamma_{kl} \\ &= r\pi \\ &= rV(t, I, \nu_1, \nu_2, \nu_3). \end{aligned}$$

where r is a risk free interest rate. As a result we can take the following lemma by Feynman-Kač Theorems.

THEOREM 4.2.2. Suppose that the stochastic process I satisfies the following SDE

$$dI = I(t)\sigma(t, T_1, T_2) \cdot d\mathbf{W}_t$$

where

$$\sigma(t,T_1,T_2) = \nu_1\sigma_X(t,T_1) + \nu_2\sigma_Y(t,T_1) + \nu_3\sigma_X(t,T_2) + \nu_4\sigma_Y(t,T_2).$$

Furthermore ν_1, ν_2 and ν_3 are satisfy the following SDE

$$\begin{aligned} d\nu_1 &= \nu_1\left(\mu_1(t,\nu_1)dt + \theta_1(t,\nu_1)\cdot d\mathbf{W}_t\right) \\ d\nu_2 &= \nu_2\left(\mu_2(t,\nu_2)dt + \theta_2(t,\nu_2)\cdot d\mathbf{W}_t\right) \\ d\nu_3 &= \nu_3\left(\mu_3(t,\nu_3)dt + \theta_3(t,\nu_3)\cdot d\mathbf{W}_t\right). \end{aligned}$$

and the stochastic process

$$\exp\left(-\int_t^{p_2} r(u)du\right)\left(\frac{\partial V}{\partial I}I\sigma + \Sigma_{k=1}^3 \frac{\partial V}{\partial \nu_k}\nu_k\theta_k\right)$$

is in $\mathscr{L}_2$.

Then

$$V(t,I,\nu_1,\nu_2,\nu_3) = \exp\left(-\int_t^{p_2} r(u)du\right)E\left[\Phi(I(p_2))|\mathscr{F}_t\right]$$

is a solution to the boundary value problem

$$\begin{cases} \frac{\partial V}{\partial t} + \Sigma_{k=1}^3 \frac{\partial V}{\partial \nu_k}\nu_k\mu_k + \frac{1}{2}\cdot\frac{\partial^2 V}{\partial I^2}(I\|\sigma\|)^2 + \frac{1}{2}\Sigma_{k=1}^3 \frac{\partial^2 V}{\partial \nu_k^2}\ (\nu_k\|\theta_k\|)^2 & \\ \quad +\Sigma_{k=1}^3 \frac{\partial^2 V}{\partial I\partial \nu_k}I\|\sigma\|\nu_k\|\theta_k\|\beta_k + \Sigma_{k\neq l}\frac{\partial^2 V}{\partial \nu_k\partial \nu_l}\nu_k\|\theta_k\|\nu_l\|\theta_l\|\gamma_{kl} = rV(t,I,\nu_1,\nu_2,\nu_3) & \\ \lim_{t\nearrow p_2} V\left(t,I(t),\nu_1(t),\nu_2(t),\nu_3(t)\right) = V\left(p_2,I(p_2),\nu(p_2)\right). & \end{cases}$$

Backward iterating the application of Theorem 4.2.1 and Theorem 4.2.2, we can get the values of the options at any time.

Chapter 5

Conclusion

The excess returns of RICI are defined as the rates of changes of sum of weighted futures prices. From this definition we tried to model commodity index in continuous version. As a result the dynamics of commodity index are a martingale under risk neutral measure and the volatilities are represented as stochastic processes defined by the prices of underlying futures. Namely, this model is a stochastic volatility model. From these dynamics we can derive Black-Scholes PDEs for European commodity index options. So, the prices of commodity index options are represented as the expectation of their contingent claims. And we can get the formulas for the hedging portfolio consisting of the underlying futures.

This thesis introduced the theoretical model for commodity index in Chapter 2. This modeling is constructed on hypothesis that our commodity index is rebalanced continuously. As a result the dynamics of our commodity index are martingale and lognormal processes under risk neutral measure. However this modeling shows that if the volatilities of the underlying futures are deterministic then the volatility of our commodity index is also derterministic. Therefore using this model, the mathematical description for commodity index is simpler and handling of commodity index modeling is easier than the stochastic volatility model. From this modeling we can take the Black-Scholes PDEs and the formulas for the hedging portfolio consisting of the underlying futures. Furthermore we can get the pricing formulas for the values of commodity index options.

Of course the stochastic processes of commodity indices move differently in the two models. However the correlation for the return rates of two distinct commodity indices in the two models come near to 1 in our simulations. Moreover Hong(2012)[19] shows that the difference of the prices of commodity index options in the two models is very slight through the simulations.

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국문초록

이 논문은 원자재 지수의 두 가지 모델을 제시한다.

첫번째 모델은 지수의 바탕이 되는 각 선물 가격의 비율을 연속적으로 리밸런싱 한다는 가정 하에서 만든 모델이다. 이 모델은 원자재 지수가 위험중립측도 하에서 마팅게일 과정이며 대수정규 분포로 표현된다는 것을 보여준다. 이 모델에서는 원자재 지수 값에 대해 특정 원자재 선물이 차지하는 비율이 항상 고유의 지수 비율을 만족한다. 이 모델로부터 유러피안 지수 옵션의 가격에 대한 편미분 방정식과 가격 공식을 얻어낼 수 있다.

두번째 모델은 지수의 바탕이 되는 각 선물 가격의 비율을 주기적으로 리밸런싱한다는 가정 하에서 만든 모델이다. 이 모델은 전형적인 확률 변동성 모형의 형태이다. 이 모델을 이용하여 유러피안 지수 옵션의 가격에 대해, 롤오버 기간과 롤오버를 하지 않는 기간에 의해 구분된 시구간별 편미분 방정식을 얻어낼 수 있다.

핵심 어휘: 원자재 지수, 원자재 선물, 원자재 지수 모델, 원자재 지수옵션, 지수옵션 가격, 헷징

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